Section 2: Integration by substitution

Exercise level 3

1. Without carrying out the integrations, put the following in order of size:

$$\int_0^1 x\,\mathrm{d}x \qquad \int_0^1 \sqrt{x}\,\mathrm{d}x \qquad \int_0^1 (2^x - 1)\,\mathrm{d}x$$

2. Let $I = \int \dfrac{\sin x}{\cos x + \sin x}\,\mathrm{d}x$.

   (i) Given $\sin x = A(-\sin x + \cos x) + B(\cos x + \sin x)$, find the values of the constants A and B.

   (ii) Using (i), split I into two integrals, and hence show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x = \frac{\pi}{4}.$$

3. Find $\displaystyle\sum_1^4 \left(\int_0^{\pi/4} (\sin^n x + \cos^n x)\mathrm{d}x \right)$.

4. (i) A function $\mathrm{f}(x)$ satisfies $\mathrm{f}(x) = \mathrm{f}(\pi - x)$ for all x. Explain what this tells you about the graph of $y = \mathrm{f}(x)$.

   (ii) Let $I = \int_0^{\pi} x\mathrm{f}(x)\mathrm{d}x$. Using the substitution $y = \pi - x$, show that

   $I = \pi \int_0^{\pi} \mathrm{f}(y)\mathrm{d}y - I$. Hence show that $I = \dfrac{\pi}{2} \int_0^{\pi} \mathrm{f}(x)\mathrm{d}x$.

   (iii) Use (ii) to find $\int_0^{\pi} x \sin x\,\mathrm{d}x$ and $\int_0^{\pi} x \sin^3 x\,\mathrm{d}x$.