Section 1: Methods of proof

Exercise level 2

1. Prove that for real numbers, $|a + b|$ is less than or equal to $|a| + |b|$, by considering each of the cases:
   (i) a and b are both positive
   (ii) a and b are both negative
   (iii) a is positive, b is negative and $|a| > |b|$
   (iv) a is positive, b is negative and $|a| < |b|$
   (v) a is negative, b is positive and $|a| > |b|$
   (vi) a is negative, b is positive and $|a| < |b|$.

2. The number 100...01 (with $3n - 1$ zeros, where n is an integer larger then 0) can be written as $10^{3n} + 1$.
   (i) Use the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ with $a = 10^n$ and $b = 1$ to write this number as the product of two factors.
   (ii) Prove that no numbers of this form are prime numbers, by showing that both factors are greater than 1 for all values of n greater than 0.

3. Prove that there are no rational number solutions to the equation $x^3 + x + 1 = 0$ using the method of proof by contradiction as follows:
   (i) Assume that there is a rational number $\frac{p}{q}$ in its lowest terms, with $p, q \neq 0$, which satisfies the equation. Substitute $x = \frac{p}{q}$ and multiply through by q^3 to obtain an equation involving p and q.
   (ii) Show that in each of the cases
      (a) p and q are both odd
      (b) p is even and q is odd
      (c) p is odd and q is even
   the left-hand side of the equation must be odd and therefore cannot be zero.
   (iii) Explain why the case p and q are both even contradicts the assumption in (i).

4. Use the method of Question 3 to prove that there is no rational number solution to the equation
$$x^5 + x^4 + x^3 + x^2 + 1 = 0$$

5. (i) Show that $n^7 - n = n(n-1)(n+1)(n^2+n+1)(n^2-n+1)$.
   (ii) Prove that if n is a positive integer then $n^7 - n$ is divisible by 7 using proof by exhaustion by considering the cases $n = 7x$, $n = 7x+1$, … $n = 7x+6$ where x is a positive integer.