Section 4: Finding distances

Section test

1. Find the distance of the point (5, 1, 2) from the plane $\mathbf{r}.\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = 11$.

2. Find the shortest distance from the point (4, 2, 2) to the line $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

3. Find the shortest distance between the two skew lines $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$ and

   $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

4. Find the shortest distance from the point P (4, 1, 6) to the plane $2x - 3y + 3z = 1$.
   Find the coordinates of the point on the plane that is closest to P.

5. Find the coordinates of the foot of the perpendicular from the point (3, 1, -1) to the line $\frac{x-1}{1} = \frac{y+6}{2} = \frac{z+2}{2}$.
   Find the distance of the point from the line.

6. Two lines have equations

   $l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 7 \\ 9 \\ -13 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$

   Points P and Q are on l_1 and l_2 respectively, such that PQ is perpendicular to both l_1 and l_2.
   Find the coordinates of P and Q.
   Find the distance PQ.

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Section test solutions

1. The shortest distance from the point $P(x_1, y_1, z_1)$ to the plane

$ax + by + cz + d = 0$ is given by $\left|\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

So the shortest distance from the point (5, 1, 2) to the plane

$3x - 4y + 5z - 11 = 0$ is given by $\left|\dfrac{15 - 4 + 10 - 11}{\sqrt{3^2 + 4^2 + 5^2}}\right| = \dfrac{10}{\sqrt{50}} = \dfrac{10}{5\sqrt{2}} = \sqrt{2}$

2. Let M be the point on the line closest to P.

$$\overrightarrow{OM} = \begin{pmatrix} 3+\lambda \\ 1-\lambda \\ -1+2\lambda \end{pmatrix}$$

$$\overrightarrow{PM} = \begin{pmatrix} 3+\lambda \\ 1-\lambda \\ -1+2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ -1-\lambda \\ -3+2\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ is perpendicular to $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, so $\begin{pmatrix} -1+\lambda \\ -1-\lambda \\ -3+2\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$

$$-1 + \lambda - (-1 - \lambda) + 2(-3 + 2\lambda) = 0$$
$$6\lambda = 6$$
$$\lambda = 1$$

So $\overrightarrow{PM} = \begin{pmatrix} -1+1 \\ -1-1 \\ -3+2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

$\left|\overrightarrow{PM}\right| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$

3. Let P and Q be the points on the two lines that are closest to each other.

P has position vector $\underset{\sim}{r} = \begin{pmatrix} 3+\lambda \\ -2+\lambda \\ -1-4\lambda \end{pmatrix}$

Q has position vector $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1+\mu \\ 3-2\mu \end{pmatrix}$

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$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 1+\mu \\ 3-2\mu \end{pmatrix} - \begin{pmatrix} 3+\lambda \\ -2+\lambda \\ -1-4\lambda \end{pmatrix} = \begin{pmatrix} -2-\lambda \\ 3+\mu-\lambda \\ 4-2\mu+4\lambda \end{pmatrix}$$

$\overrightarrow{PQ}$ is perpendicular to both lines

so $\begin{pmatrix} -2-\lambda \\ 3+\mu-\lambda \\ 4-2\mu+4\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = 0$

$$-2-\lambda+3+\mu-\lambda-4(4-2\mu+4\lambda)=0$$
$$-18\lambda+9\mu=15$$
$$-6\lambda+3\mu=5$$

and $\begin{pmatrix} -2-\lambda \\ 3+\mu-\lambda \\ 4-2\mu+4\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = 0$

$$3+\mu-\lambda-2(4-2\mu+4\lambda)=0$$
$$-9\lambda+5\mu=5$$

Solving these equations simultaneously gives $\mu=-5, \lambda=-\frac{10}{3}$

$$\overrightarrow{PQ} = \begin{pmatrix} -2+\frac{10}{3} \\ 3-5+\frac{10}{3} \\ 4+10-\frac{40}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \frac{2}{3}\sqrt{2^2+2^2+1^2} = \frac{2}{3}\times 3 = 2$$

4. The shortest distance from the point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is given by $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.

So the shortest distance from the point (4, 1, 6) to the plane $2x-3y+3z=1$ is given by $\left|\frac{8-3+18-1}{\sqrt{2^2+3^2+3^2}}\right| = \frac{22}{\sqrt{22}} = \sqrt{22}$

The line that goes through P perpendicular to the plane is

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4+2\lambda \\ 1-3\lambda \\ 6+3\lambda \end{pmatrix}$$

Substituting into the equation of the plane:

$2(4+2\lambda)-3(1-3\lambda)+3(6+3\lambda)+6=1$

$22\lambda=-22$

$\lambda=-1$

Position vector of point on plane is $\underset{\sim}{r}=\begin{pmatrix}2\\4\\3\end{pmatrix}$

So the coordinates of the point closest to P are (2, 4, 3).

5. The line has equation $\underset{\sim}{r}=\begin{pmatrix}1\\-6\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$

Let the foot of the perpendicular be M

$\overrightarrow{OM}=\begin{pmatrix}1+\lambda\\-6+2\lambda\\-2+2\lambda\end{pmatrix}$

P is the point (3, 1, -1)

$\overrightarrow{PM}=\begin{pmatrix}1+\lambda\\-6+2\lambda\\-2+2\lambda\end{pmatrix}-\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}-2+\lambda\\-7+2\lambda\\-1+2\lambda\end{pmatrix}$

This vector is perpendicular to the direction vector $\begin{pmatrix}1\\2\\2\end{pmatrix}$ for the line.

$\begin{pmatrix}-2+\lambda\\-7+2\lambda\\-1+2\lambda\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix}=0$

$-2+\lambda-14+4\lambda-2+4\lambda=0$

$9\lambda=18$

$\lambda=2$

$\overrightarrow{OM}=\begin{pmatrix}1+2\\-6+4\\-2+4\end{pmatrix}$

The coordinates of the foot of the perpendicular are $(3,-2,2)$

The distance of the point from the line is the length of the vector PM.

$$\overrightarrow{PM} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$$

$$\left|\overrightarrow{PM}\right| = \sqrt{3^2 + 3^3} = 3\sqrt{3} = 4.24 \text{ (3 s.f.)}$$

6. The coordinates of P are of the form $\underline{r} = \begin{pmatrix} 4+2\lambda \\ 1 \\ 3\lambda \end{pmatrix}$

The coordinates of Q are of the form $\underline{r} = \begin{pmatrix} 7+4\mu \\ 9-\mu \\ -13+5\mu \end{pmatrix}$

$$\overrightarrow{PQ} = \begin{pmatrix} 7+4\mu \\ 9-\mu \\ -13+5\mu \end{pmatrix} - \begin{pmatrix} 4+2\lambda \\ 1 \\ 3\lambda \end{pmatrix} = \begin{pmatrix} 3+4\mu-2\lambda \\ 8-\mu \\ -13+5\mu-3\lambda \end{pmatrix}$$

PQ is perpendicular to l_1 so $\begin{pmatrix} 3+4\mu-2\lambda \\ 8-\mu \\ -13+5\mu-3\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 0$

$$6+8\mu-4\lambda-39+15\mu-9\lambda=0$$
$$23\mu-13\lambda=33$$

PQ is perpendicular to l_2 so $\begin{pmatrix} 3+4\mu-2\lambda \\ 8-\mu \\ -13+5\mu-3\lambda \end{pmatrix} . \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} = 0$

$$12+16\mu-8\lambda-8+\mu-65+25\mu-15\lambda=0$$
$$42\mu-23\lambda=61$$

Solving these equations simultaneously gives $\lambda = 1, \mu = 2$.
So P is (6, 1, 3) and Q is (15, 7, -3).

$$\overrightarrow{PQ} = \begin{pmatrix} 9 \\ 6 \\ -6 \end{pmatrix} \text{ so } \left|\overrightarrow{PQ}\right| = 3\sqrt{3^2+2^2+2^2} = 12.4 \text{ (3 s.f.)}$$