Section 3: The equation of a plane

Section test

1. Which of the following are equations of a plane?

(i) $3x-2y+4z=5$

(ii) $\dfrac{x+5}{2}=y=\dfrac{z-1}{4}$

(iii) $\mathbf{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\2\end{pmatrix}$

(iv) $\mathbf{r}=\begin{pmatrix}-3\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}+\mu\begin{pmatrix}3\\2\\-1\end{pmatrix}$

2. The cartesian equation of a plane is $5x-3y+2z=8$
Which of the points $\mathrm{A}(3,1,-2)$, $\mathrm{B}(1,2,3)$, $\mathrm{C}(-2,1,3)$ and $\mathrm{D}(5,3,-4)$ lie on the plane?

3. An equation of the plane through A(-2, 1, 4) with vector $\mathbf{u}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ normal to the plane is:

(a) $-2x+y+4z+1=0$

(b) $3x-y+2z+1=0$

(c) $-2x+y+4z-1=0$

(d) $3x-y+2z-1=0$

4. The points *A* and *B* have coordinates (1, 2, 3) and (-2, 1, -5) respectively.
The line *AB* is perpendicular to the plane π.
The point *A* lies on the plane π.
Find the equation of the plane π in cartesian form.

5. Find the cartesian equation of the plane through the points A(-1, 0, 5), B and C given

$$\overrightarrow{\mathrm{AB}}\cdot\begin{pmatrix}2\\4\\-1\end{pmatrix}=\overrightarrow{\mathrm{AC}}\cdot\begin{pmatrix}2\\4\\-1\end{pmatrix}=0.$$

6. The point (2, 3, *k*) lies on the plane

$$\mathbf{r}=\begin{pmatrix}-2\\1\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}.$$

What is the value of *k*?

7. The equation of a plane is $\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-4\\2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

Find the Cartesian equation of this plane.

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8. Find the angle between the planes $3y - z = 4$ and $x + y - 2z = 465$.

9. Find the coordinates of the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and the plane $4x - 3y + 2z = 8$.
Find also the angle between the line and the plane.

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Section test solutions

1. Options (ii) and (iii) are equations of lines in 3 dimensions. One is written in Cartesian form, the other in vector form.
   The only two that are equations of planes are (i) and (iv).

2. Substitute each point into the equation of the plane:

   A: $5\times3-3\times1+2\times(-2)=8$

   B: $5\times1-3\times2+2\times3=5\neq8$

   C: $5\times(-2)-3\times1+2\times3=-7\neq8$

   D: $5\times5-3\times3+2\times(-4)=8$

   A and D are points on the plane.

3. Using $\underset{\sim}{r}.\underset{\sim}{n}=\underset{\sim}{a}.\underset{\sim}{n}$ with $\underset{\sim}{n}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\underset{\sim}{a}=\begin{pmatrix}-2\\1\\4\end{pmatrix}$

   $$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}-2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

   $$x\times3+y\times(-1)+z\times2=(-2)\times3+1\times(-1)+4\times2$$

   $$3x-y+2z=1$$

   $$3x-y+2z-1=0$$

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4. $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -8 \end{pmatrix}$ This is perpendicular to the plane.

   Using $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$ with $\underset{\sim}{n} = \begin{pmatrix} -3 \\ -1 \\ -8 \end{pmatrix}$ and $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ -8 \end{pmatrix}$$

   $$x \times (-3) + y \times (-1) + z \times (-8) = 1 \times (-3) + 2 \times (-1) + 3 \times (-8)$$

   $$-3x - y - 8z = -29$$

   $$3x + y + 8z - 29 = 0$$

5. Since both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular to vector $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$, this vector must be perpendicular to the plane.

   $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

   $$2x + 4y - z = -2 + 0 - 5$$

   $$2x + 4y - z + 7 = 0$$

6. Substituting $\underline{r} = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$ into the equation:

   $$\begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

   $$= \begin{pmatrix} -2 + 2\lambda + \mu \\ 1 - \mu \\ 6 - \lambda + 2\mu \end{pmatrix}$$

Considering middle row: $3 = 1 - \mu \quad \Rightarrow \mu = -2$

Considering top row: $2 = -2 + 2\lambda + \mu \quad \Rightarrow \lambda = 3$

Considering third row: $k = 6 - \lambda + 2\mu = 6 - 3 - 4 = -1$

7. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - 4\lambda + \mu \\ 1 + 2\lambda - \mu \\ \lambda + 2\mu \end{pmatrix}$

(1) $x = 3 - 4\lambda + \mu$

(2) $y = 1 + 2\lambda - \mu$

(3) $z = \lambda + 2\mu$

(1) + (2) $\Rightarrow x + y = 4 - 2\lambda \quad \Rightarrow \lambda = \frac{1}{2}(4 - x - y)$

Substituting into (2) $\Rightarrow y = 1 + (4 - x - y) - \mu \quad \Rightarrow \mu = 5 - x - 2y$

Substituting both these into (3) $\Rightarrow z = \frac{1}{2}(4 - x - y) + 2(5 - x - 2y)$

$\Rightarrow 2z = 4 - x - y + 20 - 4x - 8y$

$\Rightarrow 5x + 9y + 2z = 24$

8. The angle between the planes is equal to the angle between their normal vectors $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{vmatrix} 0 \\ 3 \\ -1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ -2 \end{vmatrix} \cos\theta$

$0 + 3 + 2 = \sqrt{3^2 + 1^2}\sqrt{1^2 + 1^2 + 2^2}\cos\theta$

$\cos\theta = \frac{5}{\sqrt{10}\sqrt{6}}$

$\theta = 49.8°$

9. A general point on the line has coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + \lambda \\ 4 + 3\lambda \\ -1 - 2\lambda \end{pmatrix}$

Substituting this into the plane $4x - 3y + 2z = 8$

$$4(1+\lambda)-3(4+3\lambda)+2(-1-2\lambda)=8$$
$$4+4\lambda-12-9\lambda-2-4\lambda=8$$
$$-9\lambda=18$$
$$\lambda=-2$$

The intersection point is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2 \\ 4+3\times-2 \\ -1-2\times-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

The intersection point is (-1, -2, 3).

The direction vector of the line is $\underset{\sim}{d} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

The normal vector of the plane is $\underset{\sim}{n} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

$$\underset{\sim}{d}.\underset{\sim}{n} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = 4-9-4=-9$$

$$|\underset{\sim}{d}| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$$
$$|\underset{\sim}{n}| = \sqrt{4^2+3^2+2^2} = \sqrt{29}$$

Acute angle between these two vectors is given by $\cos\theta = \dfrac{9}{\sqrt{14}\sqrt{29}}$

$$\theta = 63.5^\circ$$

Angle between line and plane is $90^\circ - 63.5^\circ = 26.5^\circ$