Section 2: Complex roots of polynomials

Section test

1. How many roots does the equation
   $x^4 + 3x^2 - 4 = 0$
   have?

2. Which of the following groups of numbers could be the roots of a polynomial equation with real coefficients?

(i)	3, 4, 5	(ii)	1 + i, 2, 4
(iii)	1 – i, i, 1	(iv)	2, 1 + i, 1 – i

3. Which of the following polynomials, when multiplied out, has real coefficients?

(a) (z – 2 – i)(z – 4)(z – 2 + i)
(b) (z – 2 – i)(z – 3)(z – 4 + 2i)
(c) (z – 3 + 4i)(z + 3 + 4i)
(d) (z – i)(z + i)(z – 1)(z + 1)(z – 2i)

4. 1 – 2i and 3 + i are two roots of a quartic equation with real coefficients. The other two roots are.

(a) 1 + 2i and 3 – i
(b) -1 + 2i and -3 – i
(c) -1 – 2i and -3 + i
(d) 1 – 2i and 3 - i

5. Find the quartic equation referred to in Question 4.

6. 2 + i is a root of $z^3 - z^2 - 7z + 15 = 0$. What is the real root of the equation?

7. The real root of $z^3 - 4z^2 + 14z - 20 = 0$ is 2. Find the other roots.

8. 1 + 2i is a root of the cubic equation $z^3 + az^2 + bz + 5 = 0$.
   Find the values of *a* and *b*.

9. –2 + i is a root of the equation $z^4 + 2z^3 - z^2 - 2z + 10 = 0$.
   Find the other roots.

10. The equation $z^4 + z^3 + 2z^2 + 4z - 8 = 0$ has two real roots.
    Find the four roots of the equation.

Edexcel AS FM Roots of polys 2 section test solns

Solutions to section test

1. This is a quartic equation, and so must have exactly four roots.

2. Any complex roots of a polynomial equation with real coefficients occur in conjugate pairs. Therefore (ii) and (iii) cannot be the roots of a polynomial equation with real coefficients, since the complex roots do not occur in conjugate pairs. (iv) has two complex roots which are a conjugate pair, and (i) has all real roots, so both of these could be the roots of a polynomial equation with real coefficients.

3. (a) is the only one in which the complex roots occur in conjugate pairs.

4. 1 – 2i is a root, so 1 + 2i is also a root.
   3 + i is a root, so 3 – i is also a root.

5. The equation is $(z-1+2i)(z-1-2i)(z-3+i)(z-3-i)=0$

$$\left((z-1)^2+4\right)\left((z-3)^2+1\right)=0$$
$$\left(z^2-2z+5\right)\left(z^2-6z+10\right)=0$$
$$z^4-8z^3+27z^2-50z+50=0$$

6. Since 2 + i is a root, 2 – i is also a root.
   The sum of the roots is 1 (using $\sum\alpha=-\frac{b}{a}$)
   so $\alpha+2+i+2-i=1$
   $\alpha=-3$
   So the real root is -3.

7. 2 is a root of the equation, so z – 2 is a factor.
   $z^3-4z^2+14z-20=0$
   $(z-2)(z^2-2z+10)=0$
   The other two roots are the roots of the quadratic equation $z^2-2z+10=0$

Edexcel AS FM Roots of polys 2 section test solns

$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$
$$= \frac{2 \pm \sqrt{-36}}{2}$$
$$= \frac{2 \pm 6i}{2}$$
$$= 1 \pm 3i$$

8. $(1+2i)^2 = 1 + 4i - 4 = -3 + 4i$

$(1+2i)^3 = (-3+4i)(1+2i) = -3 - 2i - 8 = -11 - 2i$

Substituting into $z^3 + az^2 + bz + 5 = 0$:

$-11 - 2i + a(-3+4i) + b(1+2i) + 5 = 0$

Equating real parts: $-11 - 3a + b + 5 = 0 \Rightarrow 3a - b = -6$

Equating imaginary parts: $-2 + 4a + 2b = 0 \Rightarrow 2a + b = 1$

Adding: $5a = -5 \Rightarrow a = -1, \quad b = 3$

9. $-2 + i$ is a root, so $-2 - i$ is a root

so $(z + 2 - i)(z + 2 + i)$ is a factor.

$(z + 2 - i)(z + 2 + i) = (z+2)^2 + 1$

$= z^2 + 4z + 5$

$z^4 + 2z^3 - z^2 - 2z + 10 = (z^2 + 4z + 5)(z^2 - 2z + 2)$

The other two roots are the roots of the quadratic equation $z^2 - 2z + 2 = 0$.

$$z = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 1}}{2}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2i}{2}$$
$$= 1 \pm i$$

The other roots are $-2 - i$, $1 + i$ and $1 - i$.

10. $f(z) = z^4 + z^3 + 2z^2 + 4z - 8$

$f(1) = 1 + 1 + 2 + 4 - 8 = 0$

$f(-2) = 16 - 8 + 8 - 8 - 8 = 0$

so $(z - 1)$ and $(z + 2)$ are factors.

$(z - 1)(z + 2) = z^2 + z - 2$

$z^4 + z^3 + 2z^2 + 4z - 8 = (z^2 + z - 2)(z^2 + 4)$

The roots of $z^2 + 4 = 0$ are $2i$ and $-2i$.

So the roots of the equation are 1, -2, $2i$ and $-2i$.