Section 2: Proof by induction

Section test

1. Prove by induction that $3+5+7+...+(2n+1)=n(n+2)$.

2. Prove by induction that 5^n-2^n is a multiple of 3 for all positive integers *n*.

3. $\mathbf{A}=\begin{pmatrix}4 & -1\\ 9 & -2\end{pmatrix}$

   Prove by induction that $\mathbf{A}^n=\begin{pmatrix}1+3n & -n\\ 9n & 1-3n\end{pmatrix}$ for all integer $n\geq 1$.

4. A sequence is defined by $u_1=3$ and $u_{n+1}=2u_n-1$.
   Prove by induction that $u_n=2^n+1$.

5. Prove by induction that $1\times 3+2\times 4+3\times 5+...+n(n+2)=\frac{1}{6}n(n+1)(2n+7)$.

6. Prove by induction that 7^n-3^n is a multiple of 4 for all positive integers *n*.

7. $\mathbf{A}=\begin{pmatrix}7 & 9\\ -4 & -5\end{pmatrix}$

   Prove by induction that $\mathbf{A}^n=\begin{pmatrix}1+6n & 9n\\ -4n & 1-6n\end{pmatrix}$ for all integer $n\geq 1$.

8. A sequence is defined by $u_1=1$ and $u_{n+1}=3u_n-4$.
   Prove by induction that $u_n=2-3^{n-1}$.

9. Prove by induction that $1\times 3+2\times 5+3\times 6+...+n(2n+1)=\frac{1}{6}n(n+1)(4n+5)$.

10. Prove by induction that 7^n+4^n+1 is a multiple of 6 for all positive integers *n*.

Edexcel AS FM Series 2 Section test solutions

Solutions to section test

1. Prove by induction that $3+5+7+...+(2n+1)=n(n+2)$.

   For $n=1$, LHS $=3$, RHS $=1\times 3=3$
   So true for $n=1$.

   Assume true for $n=k$, so $\sum_{r=1}^{k}(2r+1)=k(k+2)$
   For $n=k+1$,
   $$\sum_{r=1}^{k+1}(2r+1)=k(k+2)+[2(k+1)+1]$$
   $$=k^2+4k+3$$
   $$=(k+1)(k+3)$$
   $$=(k+1)((k+1)+2)$$
   So if the result is true for $n=k$, then it is true for $n=k+1$, and as it is true for $n=1$ it is true for all integer values of $n\geq 1$ by induction.

2. Prove by induction that 5^n-2^n is a multiple of 3 for all positive integers n.

   For $n=1$, $5^n-2^n=5-2=3$ which is a multiple of 3
   So true for $n=1$.

   Assume true for $n=k$, so $5^k-2^k=3p$ for some positive integer p.
   For $n=k+1$,
   $$5^{k+1}-2^{k+1}=5\times 5^k-2\times 2^k$$
   $$=5(2^k+3p)-2\times 2^k$$
   $$=3\times 2^k+15p$$
   Both terms are multiples of 3, so $5^{k+1}-2^{k+1}$ is a multiple of 3.

   So if the result is true for $n=k$, then it is true for $n=k+1$, and as it is true for $n=1$ it is true for all integer values of $n\geq 1$ by induction.

3. $A=\begin{pmatrix}4 & -1\\ 9 & -2\end{pmatrix}$

   Prove by induction that $A^n=\begin{pmatrix}1+3n & -n\\ 9n & 1-3n\end{pmatrix}$ for all integer $n\geq 1$.

   For $n=1$, $A=\begin{pmatrix}1+3 & -1\\ 9 & 1-3\end{pmatrix}=\begin{pmatrix}4 & -1\\ 9 & -2\end{pmatrix}$

So true for $n = 1$.

Assume true for $n = k$, so $A^k = \begin{pmatrix} 1+3k & -k \\ 9k & 1-3k \end{pmatrix}$

For $n = k+1$,

$$A^{k+1} = \begin{pmatrix} 1+3k & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4(1+3k)-9k & -1(1+3k)+2k \\ 36k+9(1-3k) & -9k-2(1-3k) \end{pmatrix}$$
$$= \begin{pmatrix} 3k+4 & -k-1 \\ 9k+9 & -3k-2 \end{pmatrix}$$
$$= \begin{pmatrix} 1+3(k+1) & -(k+1) \\ 9(k+1) & 1-3(k+1) \end{pmatrix}$$

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

4. A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n - 1$.
   Prove by induction that $u_n = 2^n + 1$.

   For $n = 1$, $u_1 = 2^1 + 1 = 2 + 1 = 3$.
   So true for $n = 1$.

   Assume true for $n = k$, so $u_k = 2^k + 1$
   For $n = k+1$,
   $$u_{k+1} = 2u_k - 1$$
   $$= 2(2^k + 1) - 1$$
   $$= 2^{k+1} + 2 - 1$$
   $$= 2^{k+1} + 1$$

   So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

5. Prove by induction that
   $1\times3 + 2\times4 + 3\times5 + \ldots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

   For $n = 1$, LHS $= 1\times3 = 3$, RHS $= \frac{1}{6}\times1\times2\times9 = 3$
   So true for $n = 1$.

Assume true for $n = k$, so $\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

For $n = k+1$

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+2) &= \tfrac{1}{6}k(k+1)(2k+7) + (k+1)((k+1)+2) \\ &= \tfrac{1}{6}(k+1)[k(2k+7)+6(k+3)] \\ &= \tfrac{1}{6}(k+1)(2k^2+13k+18) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+9) \\ &= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)\end{aligned}$$

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

6. Prove by induction that $7^n - 3^n$ is a multiple of 4 for all positive integers n.

For $n = 1$, $7^n - 3^n = 7 - 3 = 4$ which is a multiple of 4
So true for $n = 1$.

Assume true for $n = k$, so $7^k - 3^k = 4p$ for some positive integer p.
For $n = k+1$,

$$\begin{aligned}7^{k+1} - 3^{k+1} &= 7 \times 7^k - 3 \times 3^k \\ &= 7(3^k + 4p) - 3 \times 2^k \\ &= 4 \times 3^k + 28p\end{aligned}$$

Both terms are multiples of 4, so $7^{k+1} - 3^{k+1}$ is a multiple of 4.

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

7. $\mathbf{A} = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$

Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 1+6n & 9n \\ -4n & 1-6n \end{pmatrix}$ for all integer $n \geq 1$.

For $n = 1$, $\mathbf{A}^n = \begin{pmatrix} 1+6 & 9 \\ -4 & 1-6 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$.
So true for $n = 1$.

Assume true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 1+6k & 9k \\ -4k & 1-6k \end{pmatrix}$

For $n = k+1$

$$A^{k+1} = \begin{pmatrix} 1+6k & 9k \\ -4k & 1-6k \end{pmatrix}\begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$$
$$= \begin{pmatrix} 7(1+6k)-36k & 9(1+6k)-45k \\ -28k-4(1-6k) & -36k-5(1-6k) \end{pmatrix}$$
$$= \begin{pmatrix} 6k+7 & 9k+9 \\ -4k-4 & -6k-5 \end{pmatrix}$$
$$= \begin{pmatrix} 1+6(k+1) & 9(k+1) \\ -4(k+1) & 1-6(k+1) \end{pmatrix}$$

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

8. A sequence is defined by $u_1 = 1$ and $u_{n+1} = 3u_n - 4$.
   Prove by induction that $u_n = 2 - 3^{n-1}$.

For $n = 1$, $u_1 = 2 - 3^0 = 2 - 1 = 1$.
So true for $n = 1$.

Assume true for $n = k$, so $u_k = 2 - 3^{k-1}$
For $n = k+1$,

$$u_{k+1} = 3(2-3^{k-1}) - 4$$
$$= 6 - 3^k - 4$$
$$= 2 - 3^{k+1-1}$$

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

9. Prove by induction that
   $1\times 3 + 2\times 5 + 3\times 6 + \ldots + n(2n+1) = \frac{1}{6}n(n+1)(4n+5)$.

For $n = 1$, LHS $= 1\times 3 = 3$, RHS $= \frac{1}{6}\times 1\times 2\times 9 = 3$
So true for $n = 1$.

Assume true for $n = k$, so $\sum_{r=1}^{k} r(2r+1) = \frac{1}{6}k(k+1)(4k+5)$

For $n = k+1$,

$$\sum_{r=1}^{k+1} r(2r+1) = \tfrac{1}{6}k(k+1)(4k+5) + (k+1)(2(k+1)+1)$$
$$= \tfrac{1}{6}(k+1)[k(4k+5) + 6(k+1)(2k+3)]$$
$$= \tfrac{1}{6}(k+1)(4k^2+17k+18)$$
$$= \tfrac{1}{6}(k+1)(k+2)(4k+9)$$
$$= \tfrac{1}{6}(k+1)(k+2)(4(k+1)+5)$$

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.

10. Prove by induction that $7^n + 4^n + 1$ is a multiple of 6 for all positive integers n.

For $n = 1$, $7^n + 4^n + 1 = 7 + 4 + 1 = 12$ which is a multiple of 6
So true for $n = 1$.

Assume true for $n = k$, so $7^k + 4^k + 1 = 6p$ for some positive integer p.
For $n = k+1$,

$$7^{k+1} + 4^{k+1} + 1 = 7 \times 7^k + 4 \times 4^k + 1$$
$$= 7(6p - 1 - 4^k) + 4 \times 4^k + 1$$
$$= 42p - 6 - 3 \times 4^k$$

Since 4^k is even, all three terms are multiples of 6, so $7^{k+1} + 4^{k+1} + 1$ is a multiple of 6.

So if the result is true for $n = k$, then it is true for $n = k+1$, and as it is true for $n = 1$ it is true for all integer values of $n \geq 1$ by induction.