Edexcel AS Further Mathematics Matrices

Section 3: Invariance

Section test

1. Which of the following points are invariant under the transformation $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}$?

   (a) (1, -2)
   (b) (-1, 2)
   (c) (-2, 1)
   (d) (2, -1)
   (e) (0, 0)

2. For the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which of the following statements are true?

   (a) The x-axis is a line of invariant points
   (b) The x-axis is an invariant line
   (c) The y-axis is a line of invariant points
   (d) The y-axis is an invariant line
   (e) All lines of the form $x = k$ are invariant
   (f) All lines of the form $y = k$ are invariant

3. For an enlargement, centre the origin, which of the following statements are true?

   (a) All lines of the form $y = mx$ are lines of invariant points
   (b) All lines of the form $y = mx$ are invariant lines
   (c) The only invariant lines are the coordinate axes
   (d) The only invariant point is the origin

4. For the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which of the following statements are true?

   (a) The x-axis is an invariant line
   (b) The x-axis is a line of invariant points
   (c) All lines of the form $x = k$ are invariant lines
   (d) All lines of the form $y = k$ are invariant lines
   (e) The only invariant point is the origin

5. The invariant points under the transformation $\begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}$ are

   (a) all points on the line $2x + 3y = 0$
   (b) all points on the line $x + 3y = 0$
   (c) all points on the line $2x + 7y = 0$
   (d) (0, 0) only

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6. The invariant points under the transformation $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ are

   (a) (0, 0) and ($\frac{1}{2}$, 1)
   (b) (0, 0) and (1, 2)
   (c) all points on the line $y = 2x$
   (d) (0, 0) only

7. The invariant points under the transformation $\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}$ are

(a) $(2k, -k)$
(b) $(2k, k)$
(c) $(k, -2k)$
(d) $(k, 2k)$
(e) I don't know

8. The matrix $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ represents a reflection.

   The equation of the mirror line is

(a) $x = \sqrt{3}y$
(b) $y = -\sqrt{3}x$
(c) $y = \sqrt{3}x$
(d) $x = -\sqrt{3}y$
(e) I don't know

9. For the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, there are two invariant lines. One is the line $y = -x$, and the other is of the form $y = kx$. What is the value of k?

10. For the matrix $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, which of the following are invariant lines?

    (a) all lines of the form $y = x + c$
    (b) the line $y = x$ but no other lines of the form $y = x + c$
    (c) all lines of the form $y = 4x + c$
    (d) the line $y = 4x$ but no other lines of the form $y = 4x + c$

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Solutions to section test

1. $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

   $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

   $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \end{pmatrix}$

   $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

   $\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   So (1, -2), (-1, 2) and (0,0) are invariant points.

2. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reflection in the x-axis.

   So the x-axis is a line of invariant points, and is also an invariant line (since a line of invariant points is a special case of an invariant line).
   The y-axis is an invariant line, and all lines parallel to the y-axis (i.e. lines of the form $x = k$) are invariant.
   So statements (a), (b), (d) and (e) are true.

3. For an enlargement, all points on a line $y = mx$ map to another point on the same line. So all lines of this form are invariant lines (but not lines of invariant points).
   The only invariant point is the origin.
   So statements (b) and (d) are true

4. For the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the image of the point (x, y) is $(x + y, y)$. So all points on the x-axis (i.e. $y = 0$) are mapped to themselves. So the x-axis is a line of invariant points (and is therefore also an invariant line). All other points are moved parallel to the x-axis, so all lines of the form $y = k$ are invariant lines.
   So statements (a), (b) and (d) are true.

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5. $\begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x+3y=x \\ 2x+7y=y \end{matrix} \quad \Rightarrow x+3y=0$

   The invariant points are all points on the line $x+3y=0$.

6. $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x=x \\ x-y=y \end{matrix}$

   $2x=x \quad \Rightarrow x=0$

   $x-y=y \quad \Rightarrow -y=y \quad \Rightarrow y=0$

   The only invariant point is (0, 0).

7. $\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x-y=x \\ -4x+3y=y \end{matrix} \quad \Rightarrow 2x=y$

   The invariant points under this transformation are $(k, 2k)$.

8. $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} -\frac{1}{2}x+\frac{\sqrt{3}}{2}y=x \\ \frac{\sqrt{3}}{2}x+\frac{1}{2}y=y \end{matrix} \quad \Rightarrow \begin{matrix} \sqrt{3}y=3x \\ \sqrt{3}x=y \end{matrix}$

   The line of invariant points is the line $y=\sqrt{3}x$, so this is the mirror line.

9. For invariant lines of the form $y=mx+c$, the point $(x, mx+c)$ is mapped to the point $(x', mx'+c)$.

   So $\begin{pmatrix} x' \\ mx'+c \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} 3x+mx+c \\ 2x+4mx+4c \end{pmatrix}$

   Substituting $x'=3x+mx+c$ into $mx'+c=2x+4mx+4c$

   gives $m(3x+mx+c)+c=2x+4mx+4c$

   $3mx+m^2x+mc+c=2x+4mx+4c$

   $x(m^2-m-2)+mc-3c=0$

   So $m^2-m-2=0$

   $(m-2)(m+1)=0$

   $m=2$ or -1

   If $m=-1$, $-c-3c=0 \Rightarrow c=0$

   so the line $y=-x$ is an invariant line.

   If $m=2$, $2c-3c=0 \Rightarrow c=0$

   so the line $y=2x$ is an invariant line

   So k = 2.

10. For invariant lines of the form $y=mx+c$, the point $(x, mx+c)$ is mapped to the point $(x', mx'+c)$.

So $\begin{pmatrix} x' \\ mx'+c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} 2x-mx-c \\ 4x-3mx-3c \end{pmatrix}$

Substituting $x' = 2x - mx - c$ into $mx' + c = 4x - 3mx - 3c$

gives $m(2x - mx - c) + c = 4x - 3mx - 3c$

$$2mx - m^2x - mc + c = 4x - 3mx - 3c$$

$$x(5m - m^2 - 4) - mc + 4c = 0$$

So $5m - m^2 - 4 = 0$

$m^2 - 5m + 4 = 0$

$(m-1)(m-4) = 0$

$m = 1$ or $m = 4$

If $m = 1$, $-c + 4c = 0 \Rightarrow c = 0$

so the line $y = x$ is an invariant line

If $m = 4$, $-4c + 4c = 0 \Rightarrow c$ can take any value

so all lines of the form $y = 4x + c$ are invariant lines.