Section 2: The inverse of a 3×3 matrix

Section test

Questions 1 to 2 refer to the matrix $\begin{pmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ -3 & -2 & 4 \end{pmatrix}$

1. What is the cofactor of the element 4?

2. What is the cofactor of the element -2?

3. Find the determinant of the matrix $\begin{pmatrix} 4 & 2 & 1 \\ 0 & -3 & 0 \\ k & 2 & 1 \end{pmatrix}$.

Questions 4 to 5 are about the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

4. The matrix of cofactors is

(a) $\begin{pmatrix} -5 & 3 & 7 \\ 0 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -5 & -3 & 7 \\ 0 & 2 & -3 \\ 0 & -1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 0 & 0 \\ -3 & 2 & -1 \\ 7 & -3 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -5 & 0 & 0 \\ 3 & 2 & 1 \\ 7 & 3 & -1 \end{pmatrix}$

5. The inverse matrix is

(a) $\frac{1}{5}\begin{pmatrix} -5 & 3 & 7 \\ 0 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ (b) $\frac{1}{5}\begin{pmatrix} -5 & 0 & 0 \\ -3 & 2 & -1 \\ 7 & -3 & -1 \end{pmatrix}$

(c) $\frac{1}{5}\begin{pmatrix} 5 & 3 & -7 \\ 0 & -2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ (d) $\frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 3 & -2 & 1 \\ -7 & 3 & 1 \end{pmatrix}$

Edexcel AS FM Inverse matrices 2 section test solns

Questions 6 to 7 are about the matrix $\begin{pmatrix} k & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

6. The matrix of cofactors is

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 3 & k-1 & -k-2 \\ -1 & 0 & k \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -3 & -1 \\ 0 & k-1 & 0 \\ -1 & k+2 & k \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & -1 \\ -3 & k-1 & k+2 \\ -1 & 0 & k \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 & -1 \\ 0 & k-1 & 0 \\ -1 & -k-2 & k \end{pmatrix}$

7. The inverse matrix is

(a) $\frac{1}{k-1}\begin{pmatrix} 1 & 0 & -1 \\ 3 & k-1 & -k-2 \\ -1 & 0 & k \end{pmatrix}$

(b) $(k-1)\begin{pmatrix} 1 & -3 & -1 \\ 0 & k-1 & 0 \\ -1 & k+2 & k \end{pmatrix}$

(c) $\frac{1}{k-1}\begin{pmatrix} 1 & -3 & -1 \\ 0 & k-1 & 0 \\ -1 & k+2 & k \end{pmatrix}$

(d) $(k-1)\begin{pmatrix} 1 & 0 & -1 \\ 3 & k-1 & -k-2 \\ -1 & 0 & k \end{pmatrix}$

Questions 8 and 9 are about the matrix $\begin{pmatrix} x & x & 1 \\ 5 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$

8. The matrix of cofactors is

(a) $\begin{pmatrix} 1 & 1-x & 2x-3 \\ 1 & x-2 & 2x-5 \\ -1 & -x & -2x \end{pmatrix}$

(b) $\begin{pmatrix} 1 & x-1 & 2x-3 \\ -1 & x-2 & 5-2x \\ -1 & x & -2x \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & -1 \\ 1-x & x-2 & -x \\ 2x-3 & 2x-5 & -2x \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & -1 \\ 1-x & x-2 & x \\ 2x-3 & 5-2x & -2x \end{pmatrix}$

9. The inverse matrix is

(a) $\begin{pmatrix} -1 & -1 & 1 \\ x-1 & 2-x & x \\ 3-2x & 5-2x & 2x \end{pmatrix}$

(b) $\begin{pmatrix} 1 & x-1 & 2x-3 \\ -1 & x-2 & 5-2x \\ -1 & x & -2x \end{pmatrix}$

(c) $\begin{pmatrix} -1 & x-1 & 3-2x \\ 1 & 2-x & 2x-5 \\ 1 & -x & 2x \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & -1 \\ 1-x & x-2 & -x \\ 2x-3 & 2x-5 & -2x \end{pmatrix}$

10. For which of the following values of k is the matrix $\begin{pmatrix} k & 1 & 0 \\ 1 & k+1 & -1 \\ 1 & -2 & k+2 \end{pmatrix}$ singular?

(a) 1
(b) -1
(c) 2
(d) -2
(e) 3
(f) -3

Solutions to section test

1.

$$\begin{pmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ -3 & -2 & 4 \end{pmatrix}$$

The minor of the element 4 is $= \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$

$$= (2 \times 3) - (1 \times -4)$$
$$= 6 + 4$$
$$= 10$$

The place sign of the element 4 is +
so the cofactor of the element 4 is 10.

2.

$$\begin{pmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ -3 & -2 & 4 \end{pmatrix}$$

The minor of the element -2 is $= \begin{vmatrix} 2 & 0 \\ -4 & -1 \end{vmatrix}$

$$= (2 \times -1) - (0 \times -4)$$
$$= -2 + 0$$
$$= -2$$

The place sign of the element -2 is -
so the cofactor of the element -2 is 2.

3. Expanding the determinant by the second row:

$$\begin{vmatrix} 4 & 2 & 1 \\ 0 & -3 & 0 \\ k & 2 & 1 \end{vmatrix} = -3\begin{vmatrix} 4 & 1 \\ k & 1 \end{vmatrix}$$
$$= -3\left((4 \times 1) - (1 \times k)\right)$$
$$= -3(4 - k)$$
$$= 3k - 12$$

Since there are two zeros in the second row, expanding by this row makes the calculations easier.

4. $A_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = (-1\times 2)-(1\times 3) = -2-3 = -5$

$A_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -(2\times 2)+(1\times 1) = -4+1 = -3$

$A_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = (2\times 3)-(-1\times 1) = 6+1 = 7$

$A_{21} = -\begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} = -(0\times 2)+(0\times 3) = 0$

$A_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = (1\times 2)-(0\times 1) = 2-0 = 2$

$A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = -(1\times 3)+(0\times 1) = -3+0 = -3$

$A_{31} = -\begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} = -(0\times 2)+(0\times 3) = 0$

$A_{32} = -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -(1\times 1)+(0\times 2) = -1+0 = -1$

$A_{33} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (1\times -1)-(0\times 2) = -1-0 = -1$

The matrix of cofactors is $\begin{pmatrix} -5 & -3 & 7 \\ 0 & 2 & -3 \\ 0 & -1 & -1 \end{pmatrix}$

5. Expanding by first row:

$\begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -5$

The adjugate matrix is the transpose of the matrix of cofactors $\begin{pmatrix} -5 & -3 & 7 \\ 0 & 2 & -3 \\ 0 & -1 & -1 \end{pmatrix}$

The adjugate matrix is $\begin{pmatrix} -5 & 0 & 0 \\ -3 & 2 & -1 \\ 7 & -3 & -1 \end{pmatrix}$

The inverse matrix is $-\frac{1}{5}\begin{pmatrix} -5 & 0 & 0 \\ -3 & 2 & -1 \\ 7 & -3 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 3 & -2 & 1 \\ -7 & 3 & 1 \end{pmatrix}$

6. $A_{11} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = (1\times 1) - (0\times -1) = 1$

$A_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = -(0\times 1) + (0\times 1) = 0$

$A_{13} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = (0\times -1) - (1\times 1) = -1$

$A_{21} = -\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -(2\times 1) + (1\times -1) = -2 - 1 = -3$

$A_{22} = \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} = (k\times 1) - (1\times 1) = k - 1$

$A_{23} = -\begin{vmatrix} k & 2 \\ 1 & -1 \end{vmatrix} = -(k\times -1) + (2\times 1) = k + 2$

$A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = (2\times 0) - (1\times 1) = -1$

$A_{32} = -\begin{vmatrix} k & 1 \\ 0 & 0 \end{vmatrix} = -(k\times 0) + (1\times 0) = 0$

$A_{33} = \begin{vmatrix} k & 1 \\ 0 & 1 \end{vmatrix} = (k\times 1) - (1\times 0) = k$

The matrix of cofactors is $\begin{pmatrix} 1 & 0 & -1 \\ -3 & k-1 & k+2 \\ -1 & 0 & k \end{pmatrix}$

7. Expanding by second row:

$\begin{vmatrix} k & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1\big((k\times 1) - (1\times 1)\big) = k - 1$

The adjugate is the transpose of the matrix of cofactors $\begin{pmatrix} 1 & 0 & -1 \\ -3 & k-1 & k+2 \\ -1 & 0 & k \end{pmatrix}$

The adjugate matrix is $\begin{pmatrix} 1 & -3 & -1 \\ 0 & k-1 & 0 \\ -1 & k+2 & k \end{pmatrix}$

The inverse matrix is $\dfrac{1}{k-1}\begin{pmatrix} 1 & -3 & -1 \\ 0 & k-1 & 0 \\ -1 & k+2 & k \end{pmatrix}$

8. $A_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (3\times1)-(2\times1) = 3-2 = 1$

$A_{12} = -\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = -(5\times1)+(2\times2) = -5+4 = -1$

$A_{13} = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = (5\times1)-(3\times2) = 5-6 = -1$

$A_{21} = -\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix} = -(x\times1)+(1\times1) = 1-x$

$A_{22} = \begin{vmatrix} x & 1 \\ 2 & 1 \end{vmatrix} = (x\times1)-(1\times2) = x-2$

$A_{23} = -\begin{vmatrix} x & x \\ 2 & 1 \end{vmatrix} = -(x\times1)+(x\times2) = -x+2x = x$

$A_{31} = \begin{vmatrix} x & 1 \\ 3 & 2 \end{vmatrix} = (x\times2)-(1\times3) = 2x-3$

$A_{32} = -\begin{vmatrix} x & 1 \\ 5 & 2 \end{vmatrix} = -(x\times2)+(1\times5) = 5-2x$

$A_{33} = \begin{vmatrix} x & x \\ 5 & 3 \end{vmatrix} = (x\times3)-(x\times5) = 3x-5x = -2x$

The matrix of cofactors is $\begin{pmatrix} 1 & -1 & -1 \\ 1-x & x-2 & x \\ 2x-3 & 5-2x & -2x \end{pmatrix}$

9. Expanding by first column:

$$\begin{vmatrix} x & x & 1 \\ 5 & 3 & 2 \\ 2 & 1 & 1 \end{vmatrix} = xA_{11} + 5A_{21} + 2A_{31}$$

$$= x\times1 + 5(1-x) + 2(2x-3)$$
$$= x+5-5x+4x-6$$
$$= -1$$

The adjugate is the transpose of the matrix of cofactors

$$\begin{pmatrix} 1 & -1 & -1 \\ 1-x & x-2 & x \\ 2x-3 & 5-2x & -2x \end{pmatrix}$$

The adjugate matrix is $\begin{pmatrix} 1 & 1-x & 2x-3 \\ -1 & x-2 & 5-2x \\ -1 & x & -2x \end{pmatrix}$

The inverse matrix is $\frac{1}{-1}\begin{pmatrix} 1 & 1-x & 2x-3 \\ -1 & x-2 & 5-2x \\ -1 & x & -2x \end{pmatrix} = \begin{pmatrix} -1 & x-1 & 3-2x \\ 1 & 2-x & 2x-5 \\ 1 & -x & 2x \end{pmatrix}$

10. $\begin{vmatrix} k & 1 & 0 \\ 1 & k+1 & -1 \\ 1 & -2 & k+2 \end{vmatrix} = k\begin{vmatrix} k+1 & -1 \\ -2 & k+2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & k+2 \end{vmatrix}$

$$= k\left[(k+1)(k+2)-(-1\times -2)\right]-1\left[1(k+2)-(-1\times 1)\right]$$
$$= k(k+1)(k+2)-2k-(k+2)-1$$
$$= k^3+3k^2-k-3$$
$$= (k-1)(k^2+4k+3)$$
$$= (k-1)(k+1)(k+3)$$

For k = 1, k = -1 and k = -3, the determinant is zero and so the matrix is singular.