Section 2: Complex roots of polynomials

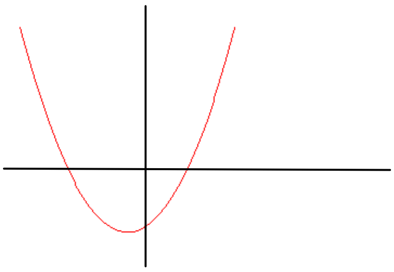
Notes and Examples

These notes contain subsections on

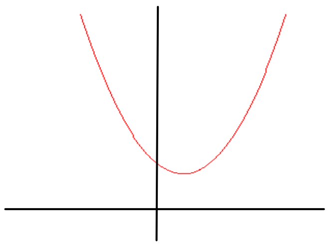
Complex numbers and equations

You now know from your work on complex numbers that every quadratic equation has exactly two solutions, if you count repeated roots and complex roots.

There are two possibilities:

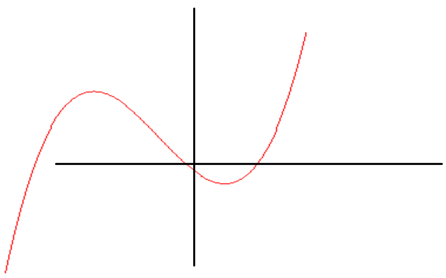


The graph crosses the x axis twice. There are two real distinct roots. If the graph just touches the axis, then the root is repeated, but this still counts as two roots.

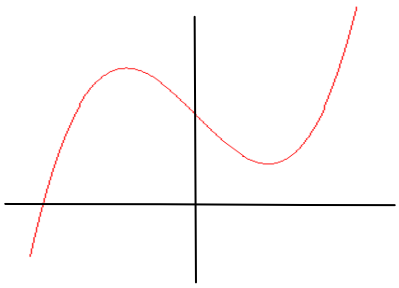


The graph does not cut the x axis. There are two complex roots which are a conjugate pair.

For a cubic equation, there are also two possibilities:



The graph cuts the x axis three times. In the diagram there are three real distinct roots. However, two of the roots could be the same, in which case the graph would touch the axis at one of the turning points, or all three roots could be the same, in which case there would be a point of inflection on the x axis.



Here the graph cuts the x axis only once. There is one real root, and there is also a conjugate pair of complex roots.

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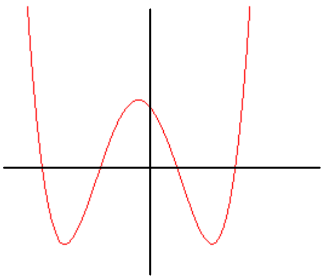
You can see graphically that a cubic equation must have at least one real root. If the term in x^3 is positive, then for large positive values of x the value of the function is large and positive, and for large negative values of x the value of the function is large and negative. (If the term in x^3 is negative, this is reversed). So all cubic graphs must cut the x axis at least once.

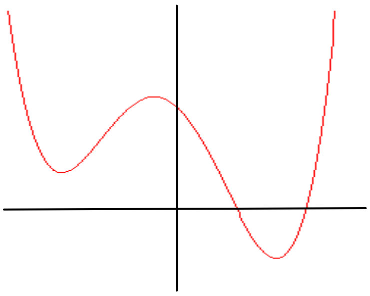
Of course, the real root may not be an integer or even a rational number, so you may not be able to find it! However, any cubic equations you meet in this section will have a simple real root, so that you can solve it.

A formula does exist for solving all cubic equations, but it is extremely complicated.

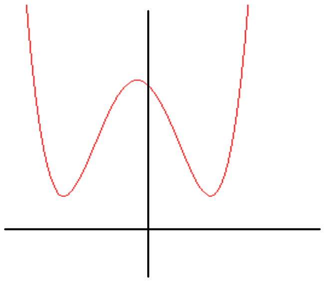
For quartic equations, there are three possibilities:



There may be four real roots, some of which may be repeated.



There may be two real roots, and a conjugate pair of complex roots.



There may be no real roots. In this case, there are two conjugate pairs of complex roots.

There is, again, an extremely complicated formula for the roots of a quartic equation.

For higher degree equations, there are no general formulae to find the roots. It is not simply that no-one has managed to find them yet: it was proved by Galois (a very interesting character) that no such formulae exist for any polynomial equations higher than quartic. If there happens to be one or more integer root which you can find by trial and error, it may be possible to solve a higher degree equation by algebraic methods. Otherwise, there are numerical methods which provide approximate solutions.

The main difficulty in proving that any polynomial equation of degree n has exactly n roots is proving that any polynomial has at least one root. If you assume that a polynomial of degree n has at least one root, then you can express the polynomial as a product of a linear factor and a polynomial of degree n-1. Then, since the assumption that any polynomial has at least one root also holds for the new polynomial of degree n-1, then you can express this polynomial as a product of a linear factor and a polynomial of degree n-2.

And so on, until the polynomial has been factorised into n linear factors, giving n roots. (This applies even if the roots are irrational or complex).

For example, if you know one root of a quintic equation, you can express it as the product of a linear factor and a quartic factor. Then since a quartic equation must have at least one root, you can express the quartic factor as the product of a linear factor and a cubic factor. Since a cubic equation has at least one root, you can express the cubic factor as the product of a linear factor and a quadratic factor, which can be factorised using the quadratic formula.

So, if we can prove that all polynomial equations have one root, then we can prove that a polynomial equation of degree n has exactly n roots using the method above. (This is an example of proof by induction, in which you show that if a statement is true for n, then it is also true for $n+1$. The proof has been stated very informally here: you will learn about proof by induction in another topic. Proving that all polynomial equations have at least one root is much more difficult: there are a number of approaches, all well beyond 'A' level. You can use a graphical approach to show that all polynomials of odd degree have at least one root, as described above for cubics, however, this is not a rigorous proof!

Solving equations with complex roots

In practice, the situations you are likely to encounter include

- cubics where you are given one complex root. In this case you can deduce a second complex root which is the conjugate of the first. You can then use the result $\sum \alpha = -\frac{b}{a}$ to find the third (real) root.

  An alternative approach is to use the two complex roots to find a quadratic factor of the cubic. You can then factorise the cubic into the quadratic factor and a linear factor (by inspection or polynomial division) and deduce the third (real) root from the linear factor. However, using the 'sum of roots' approach from above is usually much more efficient.

  See Example 1 below.

- cubics where you are given the real root (or told that an integer root exists, which you can find by trial and error). In this case you can factorise the cubic into a linear factor and a quadratic factor, by inspection or polynomial division, and then use the quadratic formula to find the other two roots.

  See Example 2 below.

- quartics where you are given a complex root. In this case you can again deduce a second complex root which is the conjugate of the first, and find a quadratic factor. You can then factorise the quartic into two quadratics, and use the quadratic formula to find the other two roots (which could be real or complex).

  See Example 3.

- quartics where you are given one or two real roots, or told that they exists. Find the real roots by trial and error if you need to, then factorise the quartic into the two known linear factors and a quadratic factor, which you can then use to find the other two roots.

Example 1
The roots of the equation $7z^3 - 8z^2 + 23z + 30 = 0$ are α, β and γ.
Given that $1 + 2\text{i}$ is a root of the equation, find the other two roots.

Solution 1
Since the equation has real coefficients, the complex pairs occur in conjugate pairs.
Hence, $1 - 2\text{i}$ is also a root of the equation.

The sum of the three roots of the equation is $\frac{8}{7}$ (using $\alpha + \beta + \gamma - \frac{b}{a}$.)

So $\alpha + 1 + 2\text{i} + 1 - 2\text{i} = \frac{8}{7}$

$$\alpha + 2 = \frac{8}{7}$$

$$\alpha = -\frac{6}{7}$$

Solution 2
Two of the roots are $1 + 2\text{i}$ and $1 - 2\text{i}$.

So a quadratic factor of the equation is
$$\begin{aligned}&\left(z-(1+2\text{i})\right)\left(z-(1-2\text{i})\right)\\ &= \left(z-1-2\text{i}\right)\left(z-1+2\text{i}\right)\\ &= (z-1)^2 - (2\text{i})^2\\ &= z^2 - 2z + 1 + 4\\ &= z^2 - 2z + 5\end{aligned}$$

So the equation is
$$7z^3 - 8z^2 + 23z + 30 = 0$$
$$(z^2 - 2z + 5)(7z + 6) = 0$$

So the third root is $-\frac{6}{7}$

Notice that Solution 1 is much quicker and easier than Solution 2. However, the technique of forming a quadratic from two complex roots is still important (see Example 3).

Example 2

The equation $z^3 - z^2 - 4z - 6 = 0$ has an integer root. Find all the roots of the equation.

Solution

Let $f(x) = z^3 - z^2 - 4z - 6$

$$f(1) = 1 - 1 - 4 - 6 = -10$$
$$f(2) = 8 - 4 - 8 - 6 = -10$$
$$f(3) = 27 - 9 - 12 - 6 = 0$$

Therefore $(z - 3)$ is a factor by the factor theorem.

$$z^3 - z^2 - 4z - 6 = 0$$
$$(z-3)(z^2 + 2z + 2) = 0$$

Here the factorising has been done by inspection, but you can use long division if you prefer.

The other roots are the roots of the quadratic equation $z^2 + 2z + 2 = 0$.
Using the quadratic formula:

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2\mathrm{i}}{2} \\ &= -1 \pm \mathrm{i} \end{aligned}$$

The roots are 3, -1 + i and -1 – i.

Example 3

Show that -2 + i is one root of the quartic equation $z^4 + 2z^3 + 2z^2 + 10z + 25 = 0$, and find the other roots.

Solution

$$z = -2 + \mathrm{i}$$
$$z^2 = (-2+\mathrm{i})^2 = 4 - 4\mathrm{i} - 1 = 3 - 4\mathrm{i}$$
$$z^3 = (3 - 4\mathrm{i})(-2+\mathrm{i}) = -6 + 11\mathrm{i} + 4 = -2 + 11\mathrm{i}$$
$$z^4 = (-2 + 11\mathrm{i})(-2 + \mathrm{i}) = 4 - 24\mathrm{i} - 11 = -7 - 24\mathrm{i}$$

Substituting into $z^4 + 2z^3 + 2z^2 + 10z + 25$:

$$\begin{aligned} &-7 - 24\mathrm{i} + 2(-2 + 11\mathrm{i}) + 2(3 - 4\mathrm{i}) + 10(-2 + \mathrm{i}) + 25 \\ &= -7 - 24\mathrm{i} - 4 + 22\mathrm{i} + 6 - 8\mathrm{i} - 20 + 10\mathrm{i} + 25 \\ &= 0 \end{aligned}$$

so -2 + i is a root of the equation.

Since -2 + i is a root, -2 – i is also a root.

Therefore (z + 2 – i) and (z + 2 + i) are both factors.

So a quadratic factor is

$$\begin{aligned}(z+2-\mathrm{i})(z+2+\mathrm{i}) &= (z+2)^2+1 \\ &= z^2+4z+4+1 \\ &= z^2+4z+5\end{aligned}$$

$$z^4+2z^3+2z^2+10z+25=0$$

$$(z^2+4z+5)(z^2-2z+5)=0$$

The other roots are the roots of the quadratic equation $z^2-2z+5=0$.

Using the quadratic formula:

$$\begin{aligned}z &= \frac{2\pm\sqrt{4-4\times1\times5}}{2} \\ &= \frac{2\pm\sqrt{-16}}{2} \\ &= \frac{2\pm4\mathrm{i}}{2} \\ &= 1\pm2\mathrm{i}\end{aligned}$$

The roots are -2 – i, -2 + i, 1 + 2i and 1 – 2i.