Section 1: Roots and coefficients

Notes and Examples

These notes contain subsections on
- Quadratic equations
- Cubic equations
- Quartic equations

Quadratic equations

If you want to find a quadratic equation whose roots are, say, 2 and -5, a simple way to do this is to write the equation in the form

$$(x-2)(x+5)=0$$

and then multiply out to give the equation

$$x^2+3x-10=0.$$

However, if the roots are complex, or if you want to find a cubic or quartic equation, this method is time consuming and it is easy to make mistakes in the algebra.

Finding relationships between the roots and coefficients of polynomial equations gives an alternative approach which is often simpler.

You can write the equation $ax^2+bx+c=0$
in the form $a(x-\alpha)(x-\beta)$
where α and β are the roots of the equation.

Multiplying out and equating coefficients gives the relationships

$$\alpha+\beta=-\frac{b}{a}$$

and $$\alpha\beta=\frac{c}{a}.$$

Alternatively, these relationships can be shown by using the quadratic formula:

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

$$\alpha\beta = \frac{\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)}{4a^2}$$

using $(x+y)(x-y) = x^2 - y^2$

$$= \frac{b^2-\left(b^2-4ac\right)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Example 1

Write down the sum and product of the roots of the following equations:

(i) $2x^2+7x+6=0$

(ii) $5x^2-x-1=0$

(iii) $x(x+8)=4-3x$

Solution

(i) a = 2, b = 7, c = 6

Sum of roots = $-\frac{b}{a} = -\frac{7}{2}$

Product of roots = $\frac{c}{a} = \frac{6}{2} = 3$

(ii) a = 5, b = -1, c = -1

Sum of roots = $-\frac{b}{a} = -\frac{-1}{5} = \frac{1}{5}$

Product of roots = $\frac{c}{a} = \frac{-1}{5} = -\frac{1}{5}$

(iii) $x^2+8x=4-3x$

$x^2+11x-4=0$

a = 1, b = 11, c = -4

Sum of roots = $-\frac{b}{a} = -\frac{11}{1} = -11$

Product of roots = $\frac{c}{a} = \frac{-4}{1} = -4$

Example 2

Find a quadratic equation with roots 5 and -3.

Solution

The sum of the roots is 2 $\Rightarrow -\frac{b}{a} = 2 \quad \Rightarrow b = -2a$

The product of the roots is -15 $\Rightarrow \frac{c}{a} = -15 \quad \Rightarrow c = -15a$

Taking a to be 1 gives b = -2 and c = -15.

A quadratic equation with roots 5 and -3 is $x^2-2x-15=0$

Notice that in Example 2, there are an infinite number of possible solutions. For example, if you take a to be 2, you obtain the equation $2x^2-4x-30=0$, which has the same roots. However, taking a to be 1 gives the simplest equation, unless either b or c turn out to be fractions, in which case you might choose a suitable value of a to make the coefficients integers. This is illustrated in Example 3.

Example 3

Find a quadratic equation with roots -2 and -0.5.

Solution

The sum of the roots is -2.5 $\quad \Rightarrow -\dfrac{b}{a}=-2.5 \quad \Rightarrow b=2.5a$

The product of the roots is 1 $\quad \Rightarrow \dfrac{c}{a}=1 \quad \Rightarrow c=a$

Taking a to be 2 gives b = 5 and c = 2.

A quadratic equation with roots -2 and -0.5 is $2x^2+5x+2=0$

An application of these relationships which you meet in this section is to find an equation whose roots are related to the roots of the original equation. One method is to find the values of the sum and product of the roots of the new equation are found. Another method is called the substitution method, and is usually easier.

Example 4 shows using the substitution method.

Example 4

The quadratic equation $x^2-3x+1=0$ has roots α and β.

Find equations with roots $2\alpha-1$ and $2\beta-1$.

Solution

Let $u=2x-1$ so $x=\dfrac{u+1}{2}$

$$x^2-3x+1=0$$

$$\left(\frac{u+1}{2}\right)^2-3\left(\frac{u+1}{2}\right)+1=0$$

$$\frac{u^2+2u+1}{4}-\frac{3u+3}{2}+1=0$$

$$u^2+2u+1-6u-6+4=0$$

$$u^2-4u-1=0$$

The new equation is $u^2-4u-1=0$

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Cubic equations

For a cubic equation $ax^3 + bx^2 + cx + d = 0$, with roots α, β and γ, the relationships between the roots and coefficients are given by

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

These relationships can be proved using the same approach as for quadratic equations, by writing the equation in the form $a(x-\alpha)(x-\beta)(x-\gamma)$, multiplying out and equating coefficients.

Example 5
Find a cubic equation with roots 2, -1 and -3.

Solution

$$\alpha + \beta + \gamma = 2 - 1 - 3 = -2$$

$$\Rightarrow -\frac{b}{a} = -2 \Rightarrow b = 2a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = (2 \times -1) + (-1 \times -3) + (-3 \times 2) = -2 + 3 - 6 = -5$$

$$\Rightarrow \frac{c}{a} = -5 \Rightarrow c = -5a$$

$$\alpha\beta\gamma = 2 \times -1 \times -3 = 6$$

$$\Rightarrow -\frac{d}{a} = 6 \Rightarrow d = -6a$$

Let $a = 1 \Rightarrow b = 2, c = -5, d = -6$

The equation is $z^3 + 2z^2 - 5z - 6 = 0$

As for quadratics, you need to be able to find cubics with roots related to those of a given cubic equation. Again, the substitution method can again be used to do this.

Quartic equations

For a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, with roots α, β, γ and δ, the relationships between the roots and coefficients are given by

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$$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$\sum \alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

Be careful in particular when working out the sum of the product of the roots in pairs, to make sure that you include all six terms!

The type of questions you will be asked on quartic equations are similar to those for cubic or quadratic, although of course they may involve a little more work!

Here are two more examples.

Example 6
Find the quartic equation with roots 1, -2, 3 (repeated).

Solution

$$\alpha + \beta + \gamma + \delta = 1 - 2 + 3 + 3 = 5$$

$$\begin{aligned}\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= (1\times -2) + (1\times 3) + (1\times 3) + (-2\times 3) + (-2\times 3) + (3\times 3)\\ &= -2 + 3 + 3 - 6 - 6 + 9\\ &= 1\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta &= (1\times -2\times 3) + (-2\times 3\times 3) + (3\times 3\times 1) + (3\times 1\times -2)\\ &= -6 - 18 + 9 - 6\\ &= -21\end{aligned}$$

$$\alpha\beta\gamma\delta = 1\times -2\times 3\times 3 = -18$$

$$-\frac{b}{a} = 5, \quad \frac{c}{a} = 1, \quad -\frac{d}{a} = -21, \quad \frac{e}{a} = -18$$

Let $a = 1 \;\Rightarrow\; b = -5,\ c = 1,\ d = 21,\ e = -18$

The equation is $z^4 - 5z^3 + z^2 + 21z - 18 = 0$

Example 7
The roots of the quartic equation $2z^4 + z^3 - 3z^2 + 2z - 4 = 0$ are $\alpha, \beta, \gamma, \delta$. Find the quartic equation with roots $2\alpha+1$, $2\beta+1$, $2\gamma+1$, $2\delta+1$.

Solution
Use the substitution method.

$$w = 2z + 1 \Rightarrow z = \frac{w-1}{2}$$

$$2z^4 + z^3 - 3z^2 + 2z - 4 = 0$$
$$2\left(\frac{w-1}{2}\right)^4 + \left(\frac{w-1}{2}\right)^3 - 3\left(\frac{w-1}{2}\right)^2 + 2\left(\frac{w-1}{2}\right) - 4 = 0$$
$$\frac{2(w^4 - 4w^3 + 6w^2 - 4w + 1)}{16} + \frac{(w^3 - 3w^2 + 3w - 1)}{8} - \frac{3(w^2 - 2w + 1)}{4} + w - 1 - 4 = 0$$
$$w^4 - 4w^3 + 6w^2 - 4w + 1 + w^3 - 3w^2 + 3w - 1 - 6(w^2 - 2w + 1) + 8(w - 5) = 0$$
$$w^4 - 4w^3 + 6w^2 - 4w + 1 + w^3 - 3w^2 + 3w - 1 - 6w^2 + 12w - 6 + 8w - 40 = 0$$
$$w^4 - 3w^3 - 3w^2 + 19w - 46 = 0$$

You should now be able to see how the pattern of relationships between the roots and coefficients of polynomial equations develops. You can probably predict the relationships for quintic (degree 5) equations, and you could of course continue this to equations of higher degree. However, you will not be asked to deal with equations of any higher degree than quartic.