Section 2: Integrating factors

Section test

1. Which of the following differential equations is **not** linear?

   (a) $\frac{dy}{dx} = 1 - xy$ (b) $\frac{1}{x}\frac{dy}{dx} - y = x + 1$

   (c) $x\frac{dy}{dx} + y = x - x^2 y$ (d) $\frac{dy}{dx} + xy^2 = e^x$

2. Which of the following differential equations is an exact differential equation?

   (a) $e^{-x^2}\frac{dy}{dx} - x^2 e^{-x^2} y = x$ (b) $e^{-x^2}\frac{dy}{dx} + 2xe^{-x^2} y = x$

   (c) $e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2} y = x$ (d) $e^{-x^2}\frac{dy}{dx} + e^{-x^2} y = x$

3. The general solution of the differential equation

   $$\sin x\frac{dy}{dx} + y\cos x = 1$$

   is given by

   (a) $y = \frac{x+c}{\sin x}$ (b) $y = \frac{x+c}{\cos x}$

   (c) $y = \frac{x}{\sin x} + c$ (d) $y = \frac{x}{\cos x} + c$

Questions 4 and 5 are about the differential equation $x\frac{dy}{dx} = 2y + x^2$.

4. The integrating factor for the differential equation is:

   (a) x^2 (b) $\frac{1}{x^2}$

   (c) $-2x$ (d) e^{-2x}

5. The general solution of the differential equation is:

   (a) $y = x^2 \ln|x| + c$ (b) $y = x^2 \ln|Ax|$

   (c) $y = x^2\left(\frac{1}{2}x^2 + c\right)$ (d) $y = \frac{1}{2}x^4 + c$

Questions 6 and 7 are about the differential equation $x\frac{dy}{dx} + (x+1)y = x$

Edexcel FM First order DEs 2 section test solutions

6. The integrating factor for the differential equation is:

(a) $e^{\frac{1}{2}x^2+x}$ (b) xe^x

(c) $x+e^x$ (d) $x\ln x$

7. The general solution of the differential equation is:

(a) $y = Ae^{-x}$ (b) $y = 1-\frac{1}{x}+c$

(c) $y = e^{-x}+c$ (d) $y = 1-\frac{1}{x}+\frac{c}{x}e^{-x}$

8. The general solution of the differential equation

$$x\frac{dy}{dx}+3y = x^2$$

is given by:

(a) $y = \frac{x}{5}+c$ (b) $y = x^2+cx$

(c) $y = \frac{x^2}{5}+\frac{c}{x^3}$ (d) $y = \frac{x^2}{2}+\frac{c}{x}$

9. The particular solution of the differential equation

$$\frac{dy}{dx}+2y = x$$

for which $y = 0$ when $x = 0$

is given by

(a) $y = 1-e^{-2x}$ (b) $y = \frac{1}{2}x^2e^{-2x}$

(c) $y = \frac{e^{-2x}+2x-1}{4}$ (d) $y = \frac{1-e^{-2x}}{2}$

10. The particular solution of the differential equation

$$(x^2+1)\frac{dy}{dx}+xy = x(x^2+1)$$

in the case where $x = 0$ when $y = 0$

is given by:

(a) $y = \frac{1}{3}(x^2+1)^{3/2}-\frac{1}{3}$ (b) $y = \frac{x^2(x^2+2)}{4(x^2+1)}$

(c) $y = \frac{1}{3}(x^2+1)-\frac{1}{3}(x^2+1)^{-\frac{1}{2}}$ (d) $y = \frac{1}{3}(x^2+1)+\frac{1}{3}(x^2+1)^{\frac{1}{2}}$

Edexcel FM First order DEs 2 section test solutions

Solutions to section test

1. $\dfrac{dy}{dx} + xy^2 = e^x$ is not linear, since it has a term in y^2.

2. $e^{-x^2}\dfrac{dy}{dx} - 2xe^{-x^2}y = x$ is an exact differential equation, since the derivative of e^{-x^2} is $-2xe^{-x^2}$, and so the differential equation can be written as
$\dfrac{d}{dx}\left(e^{-x^2}y\right) = x$.

3. This is an exact differential equation.

$$\sin x \frac{dy}{dx} + y\cos x = 1$$
$$\frac{d}{dx}\left(y\sin x\right) = 1$$
$$y\sin x = x + c$$
$$y = \frac{x+c}{\sin x}$$

4. $x\dfrac{dy}{dx} = 2y + x^2$

$$x\frac{dy}{dx} - 2y = x^2$$
$$\frac{dy}{dx} - \frac{2}{x}y = x$$

Integrating factor $= e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \dfrac{1}{x^2}$

5. Multiplying through by the integrating factor from question 4:

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \frac{1}{x}$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x}$$
$$\frac{y}{x^2} = \ln|x| + c = \ln Ax$$
$$y = x^2\ln|Ax|$$

Let $c = \ln A$

Edexcel FM First order DEs 2 section test solutions

6. $x\dfrac{dy}{dx}+(x+1)y=x$

$\dfrac{dy}{dx}+\left(\dfrac{x+1}{x}\right)y=1$

$\dfrac{dy}{dx}+\left(1+\dfrac{1}{x}\right)y=1$

Integrating factor $= e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x$

7. Multiplying through by the integrating factor from Question 6:

$xe^x\dfrac{dy}{dx}+e^x(x+1)y=xe^x$

$\dfrac{d}{dx}\left(xye^x\right)=xe^x$

$xye^x = xe^x - \int e^x dx$

Using integration by parts

$xye^x = xe^x - e^x + c$

$y = 1-\dfrac{1}{x}+\dfrac{c}{x}e^{-x}$

8. $x\dfrac{dy}{dx}+3y=x^2$

$\dfrac{dy}{dx}+\dfrac{3}{x}y=x$

Integrating factor $= e^{\int\frac{3}{x}dx} = e^{3\ln x} = x^3$

$x^3\dfrac{dy}{dx}+3x^2y=x^4$

$\dfrac{d}{dx}\left(x^3y\right)=x^4$

$x^3y=\dfrac{x^5}{5}+c$

$y=\dfrac{x^2}{5}+\dfrac{c}{x^3}$

9. $\dfrac{dy}{dx}+2y=x$

Integrating factor $= e^{\int 2dx} = e^{2x}$

Edexcel FM First order DEs 2 section test solutions

$$e^{2x}\frac{dy}{dx}+2e^{2x}y=xe^{2x}$$
$$\frac{d}{dx}\left(ye^{2x}\right)=xe^{2x}$$
$$ye^{2x}=\tfrac{1}{2}xe^{2x}-\int\tfrac{1}{2}e^{2x}dx$$
$$ye^{2x}=\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}+c$$
$$y=\tfrac{1}{2}x-\tfrac{1}{4}+ce^{-2x}$$
When $x=0,\ y=0\Rightarrow 0=-\tfrac{1}{4}+c\quad\Rightarrow c=\tfrac{1}{4}$
$$y=\tfrac{1}{2}x-\tfrac{1}{4}+\tfrac{1}{4}e^{-2x}$$
$$=\frac{e^{-2x}+2x-1}{4}$$

10. $(x^2+1)\dfrac{dy}{dx}+xy=x(x^2+1)$
$$\frac{dy}{dx}+\left(\frac{x}{x^2+1}\right)y=x$$
Integrating factor $=e^{\int\frac{x}{x^2+1}dx}=e^{\frac{1}{2}\ln(x^2+1)}=\sqrt{x^2+1}$
$$\sqrt{x^2+1}\frac{dy}{dx}+\left(\frac{x}{\sqrt{x^2+1}}\right)y=x\sqrt{x^2+1}$$
$$\frac{d}{dx}\left(y\sqrt{x^2+1}\right)=x\sqrt{x^2+1}$$
$$y\sqrt{x^2+1}=\tfrac{1}{3}\left(x^2+1\right)^{3/2}+c$$
$$y=\tfrac{1}{3}\left(x^2+1\right)+\frac{c}{\sqrt{x^2+1}}$$
When $x=0,\ y=0\Rightarrow 0=\tfrac{1}{3}+c\quad\Rightarrow c=-\tfrac{1}{3}$
$$y=\tfrac{1}{3}\left(x^2+1\right)-\tfrac{1}{3}\left(x^2+1\right)^{-1/2}$$