Section 2: Integrating factors

Notes and Examples

These notes contain subsections on:

- Exact differential equations
- Integrating factors

Exact differential equations

Notice that for an expression of the form

$$f(x)\frac{dy}{dx}+g(x)y$$

to be a perfect derivative, then $g(x)$ must be the derivative of $f(x)$.

So the expression can be written as

$$f(x)\frac{dy}{dx}+f'(x)y$$

or $$\frac{d}{dx}\big(yf(x)\big).$$

Integrating factors

An integrating factor is a method that can be used to write some first order differential equations in the form of an exact differential equation, so that they can be integrated.

If a differential equation can be written in the form

$$\frac{dy}{dx}+Py=Q$$

where P and Q are both functions of x, then an integrating factor of the form $e^{\int P(x)dx}$ is used. Each term is multiplied by the integrating factor, and the left-hand side of the equation can then be written as a single derivative.

Not all first order differential equations can be solved using an integrating factor – it must be possible to rearrange the equation into the form shown above in order to use this method.

The laws of logarithms and exponentials are often needed in simplifying the integrating factor.

e.g. $$e^{\ln x}=x$$
$$e^{a\ln x}=e^{\ln x^a}=x^a$$

If you do not simplify where possible, you will make things very difficult for yourself!

However, it is not always possible to simplify the integrating factor, and you may be left with an exponential function. Here is an example where this occurs.

Example 1

Find the particular solution of the differential equation

$$\frac{dy}{dx} + xy = x$$

for which $y = 3$ when $x = 0$.

Solution

The differential equation is already in the standard form.

Integrating factor $= e^{\int x\,dx} = e^{\frac{1}{2}x^2}$

Multiplying through by the integrating factor gives

$$e^{\frac{1}{2}x^2}\frac{dy}{dx} + xe^{\frac{1}{2}x^2}y = xe^{\frac{1}{2}x^2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right) = xe^{\frac{1}{2}x^2}$$

$$\Rightarrow e^{\frac{1}{2}x^2}y = e^{\frac{1}{2}x^2} + c$$

$$\Rightarrow y = 1 + ce^{-\frac{1}{2}x^2}$$

Substituting $x = 0$ and $y = 3$ gives

$3 = 1 + c$

$\Rightarrow c = 2$

The particular solution of the differential equation is

$$y = 1 + 2e^{-\frac{1}{2}x^2}$$