Section 1: Introduction

Section test

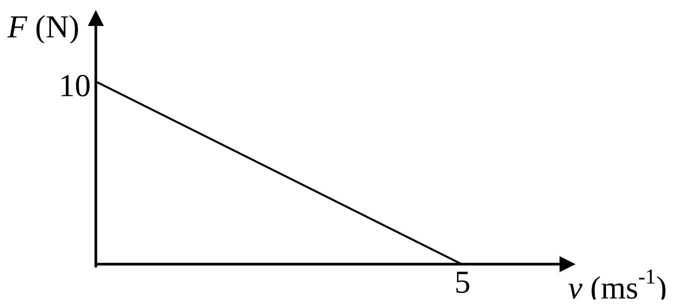
1. The concentration C of pollution in a lake decreases at a rate proportional to the concentration at any given time. Initially the concentration is 5 gm^{-3}, and the concentration is decreasing at a rate of 0.1 gm^{-3}s^{-1}.
   Find a differential equation that models this situation.

2. Water is escaping from a tank in the shape of a cube, of side 2 m, at a rate proportional to the square root of the height, h, of the water in the tank. Initially the height of water is 100 cm and water is escaping at a rate of 5 cm^{3}s^{-1}. The rate of change of h, in cms^{-1}, is given by the differential equation

   (a) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{80000}$ (b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{8}$

   (c) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{20000}$ (d) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{2}$

3. A variable force F acts on a particle of mass m.
   The graph below shows how F varies with the speed v of the particle.



   A resistive force proportional to the speed of the particle also acts on it.
   Initially the velocity of the particle is 2 ms^{-1} and its acceleration is 0 ms^{-2}.
   The motion of the particle is modelled by the differential equation

   (a) $\dfrac{dv}{dt} = \dfrac{5v-10}{m}$ (b) $\dfrac{dv}{dt} = \dfrac{10-5v}{m}$

   (c) $\dfrac{dv}{dt} = \dfrac{10-2v}{m}$ (d) $\dfrac{dv}{dt} = m(10-5v)$

4. Which of the following differential equations can be solved by separating the variables?

   (a) $\dfrac{dy}{dx} = e^{x+y}$ (b) $\dfrac{dy}{dx} = x^2 + xy$

   (c) $\dfrac{dy}{dx} = x + xy^2$ (d) $\dfrac{dy}{dx} = \dfrac{y}{y+1}$

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5. The general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 y \sin x$

is given by:

(a) $y = \arctan(c - \cos x)$ (b) $y = \arctan(\cos x + c)$

(c) $y = \arccos\left(\dfrac{1}{\cos x + c}\right)$ (d) $y = \arccos\left(\dfrac{1}{c - \cos x}\right)$

6. The particular solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 \mathrm{e}^{2x}$

for which $y = 1$ when $x = 0$ is given by

(a) $y = \dfrac{1}{2 - \mathrm{e}^{2x}}$ (b) $y = \dfrac{2}{\mathrm{e}^{2x} + 1}$

(c) $y = \dfrac{2}{3 - \mathrm{e}^{2x}}$ (d) $y = \mathrm{e}^{-2x}$

7. The general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2}{x+1}$$

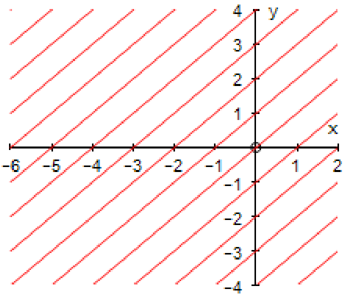
is given by

(a) $y = A(x+3)$ (b) $y = A(x+1) + 2$

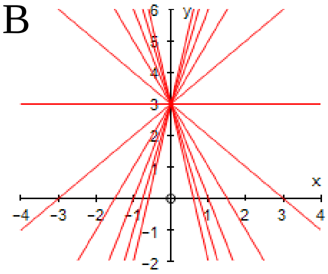
(c) $y = Ax + 3$ (d) $y = x + A$

8. The family of solution curves for the differential equation in the previous question is

A



B



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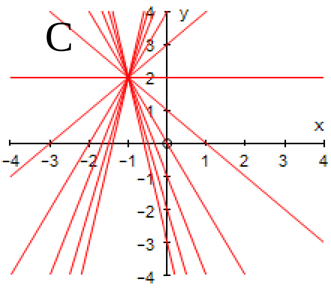

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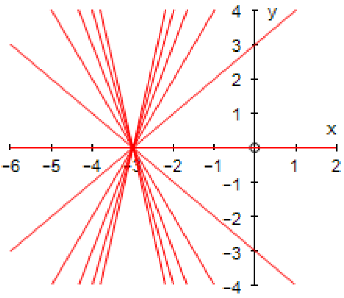

9. A drink initially at 5°C is left in a warm room.
The temperature T of the drink at time t is given by the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 3(25 - T)$$

Find the particular solution of this differential equation corresponding to the initial conditions given.

10. A population of mice increases such that its rate of increase is proportional to the size of the population.
Initially there are 20 mice, and the population is increasing at the rate of 4 mice per month.
After t months the size of the population, P, is given by

(a) $P = 20e^{5t}$ (b) $P = e^{5t} + 19$

(c) $P = 20e^{t/5}$ (d) $P = e^{t/5} + 19$

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Solutions to section test

1. $\frac{dC}{dt} = kC$ where k is a constant.

   Initially $C = 5$ and $\frac{dC}{dt} = -0.1$

   so $-0.1 = 5k \Rightarrow k = -0.02$

   Therefore $\frac{dC}{dt} = -0.02C$

2. Let the volume of water in the tank at time t be V.

   $\frac{dV}{dt} = k\sqrt{h}$

   When $h = 100$, $\frac{dV}{dt} = -5$, so $-5 = k\sqrt{100} \Rightarrow k = -0.5$

   so $\frac{dV}{dt} = -0.5\sqrt{h}$

   $V = 200 \times 200 \times h = 40000h \Rightarrow \frac{dV}{dh} = 40000$

   $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{40000} \times -0.5\sqrt{h} = -\frac{\sqrt{h}}{80000}$

3. Gradient of graph $= -2$, so equation of line is $F = 10 - 2v$

   Resistive force $= kv$

   Net force on particle $= 10 - 2v - kv$

   Newton's 2nd law: $10 - 2v - kv = m\frac{dv}{dt}$

   Initially $v = 2$ and $\frac{dv}{dt} = 0$, so $10 - 4 - 2k = 0 \Rightarrow k = 3$

   Therefore $m\frac{dv}{dt} = 10 - 2v - 3v$

   $\frac{dy}{dx} = \frac{10 - 5v}{m}$

4. For the equation $\frac{dy}{dx} = \frac{y}{y+1}$, the right-hand side is a function of y only, so the variables can be separated.

   The equation $\frac{dy}{dx} = e^{x+y}$ can be written as $\frac{dy}{dx} = e^x e^y$, which is a function of x only multiplied by a function of y only, so the variables can be separated.

   The equation $\frac{dy}{dx} = x + xy^2$ can be written as $\frac{dy}{dx} = x\left(1 + y^2\right)$, which is a function of x only multiplied by a function of y only, so the variables can be separated.

The equation $\frac{dy}{dx} = x^2 + xy$ cannot be written as a function of x only multiplied by a function of y only, so the variables cannot be separated.

5. $$\frac{dy}{dx} = \cos^2 y \sin x$$
$$\int \sec^2 y \, dy = \int \sin x \, dx$$
$$\tan y = -\cos x + c$$
$$y = \arctan(c - \cos x)$$

6. $$\frac{dy}{dx} = y^2 e^{2x}$$
$$\int \frac{1}{y^2} dy = \int e^{2x} dx$$
$$-\frac{1}{y} = \frac{1}{2}e^{2x} + c$$
When $x = 0$, $y = 1$: $\quad -1 = \frac{1}{2} + c \quad \Rightarrow c = -\frac{3}{2}$
$$-\frac{1}{y} = \frac{1}{2}e^{2x} - \frac{3}{2} = \frac{e^{2x} - 3}{2}$$
$$y = \frac{2}{3 - e^{2x}}$$

7. $$\frac{dy}{dx} = \frac{y-2}{x+1}$$
$$\int \frac{1}{y-2} dy = \int \frac{1}{x+1} dx$$
$$\ln|y-2| = \ln|x+1| + c$$
$$\ln\left|\frac{y-2}{x+1}\right| = c$$
$$\frac{y-2}{x+1} = A$$
where $A = e^c$
$$y = A(x+1) + 2$$

8. The family of lines $y = A(x+1) + 2$ are straight lines all passing through the point $(-1, 2)$.

This is graph C.

9. $\dfrac{dT}{dt} = 3(25 - T)$

$\displaystyle\int \frac{1}{25 - T}\, dT = \int 3\, dt$

$-\ln(25 - T) = 3t + c$

When $t = 0$, $T = 5$: $\quad -\ln 20 = c$

Modulus signs are not needed since $25 - T$ will never be negative

$-\ln(25 - T) = 3t - \ln 20$

$\ln(25 - T) - \ln 20 = -3t$

$\ln\left(\dfrac{25 - T}{20}\right) = -3t$

$\dfrac{25 - T}{20} = e^{-3t}$

$T = 25 - 20e^{-3t}$

10. $\dfrac{dP}{dt} = kP$

Initially $P = 20$ and $\dfrac{dP}{dt} = 4$, so $4 = 20k \quad \Rightarrow k = \frac{1}{5}$

$\dfrac{dP}{dt} = \frac{1}{5}P$

$\displaystyle\int \frac{5}{P}\, dP = \int dt$

$5\ln P = t + c$

When $t = 0$, $P = 20 \Rightarrow 5\ln 20 = c$

$5\ln P = t + 5\ln 20$

$\ln \dfrac{P}{20} = \dfrac{t}{5}$

$\dfrac{P}{20} = e^{t/5}$

$P = 20e^{t/5}$