Section 1: Introduction

Notes and Examples

These notes contain subsections on
- **Modelling with differential equations**
- **Manipulating logarithms and exponentials**
- **Using different techniques of integration**

If you have covered the work on differential equations in A level Mathematics, you will already have met the idea of forming differential equations and the method of separating the variables. In this section you will review this work before going on in the next section to learn a further method for solving first order differential equations.

Modelling with differential equations

Situations involving rates of change can often be modelled using differential equations.

For example, if you are looking at how the population, P, of a country changes over time, you would be interested in $\frac{\mathrm{d}P}{\mathrm{d}t}$, the rate of change of P with respect to t. A model for this rate of change would probably depend on many factors, including the value of P itself, so a model for $\frac{\mathrm{d}P}{\mathrm{d}t}$ would probably be a function involving both P and t. This is a differential equation.

Differential equations are often used to model situations in kinematics. You should be familiar with the following relationships:
- If x is a displacement, then $\frac{\mathrm{d}x}{\mathrm{d}t}$, or $\dot{x}$, is the rate of change of displacement, or the velocity
- If v is a velocity, then $\frac{\mathrm{d}v}{\mathrm{d}t}$, or $\dot{v}$, is the rate of change of velocity, or the acceleration

Note that the 'dot' notation is used for derivatives with respect to time only.

The expression $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ can be rewritten as $a = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ using the chain rule. Since $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, this means that another expression for acceleration is $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$. This can be useful in situations where you are interested in the relationship between velocity and distance, without reference to time.

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Manipulating logarithms and exponentials

Differential equations with separable variables often involve integrations of the form

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

or $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

You may be able to leave out the modulus signs if you are given appropriate information. For example, if you are given that $x > 0$, then you can write

$\int \frac{1}{x}\,dx = \ln x + c$, with no modulus signs.

You may need to use the rules of logarithms to rearrange expressions:

$\log a + \log b = \log ab$
$\log a - \log b = \log a/b$
$\log a^n = n \log a$

You will also need to use the fact that e^x is the inverse function of $\ln x$.

$\ln x = a \Rightarrow x = e^a$

When working with exponentials and logarithms, you may need to replace the constant of integration with a new constant related to the original one, to make the manipulation easier.

Example 1 shows how this is done.

Example 1
Find the general solution of the differential equation
$\frac{dy}{dx} = \frac{y+1}{x}$, where $x, y > 0$

Solution

$$\frac{dy}{dx} = \frac{y+1}{x}$$

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x}\,dx$$

$$\ln(y+1) = \ln x + c$$

Since both x and y are positive, no modulus signs are needed.

$$\ln(y+1) - \ln x = c$$

$$\ln\left(\frac{y+1}{x}\right) = c$$

$$\frac{y+1}{x} = e^c = A$$
$$y+1 = Ax$$
$$y = Ax - 1$$

Replace e^c with a new constant A.

Note that in the example above, the third line gives the solution to the differential equation. The rest of the working involves rearranging the solution to give y in terms of x. You may not always need to do this. However, if you are asked to give y in terms of x, you may need to use the sort of techniques shown in Example 1.

Using different techniques of integration

In some of the exercise questions, you will need to integrate a variety of functions, such as trigonometric functions and exponential functions, and use several different integration techniques which you have met in A level Mathematics, such as integration by parts, integration by substitution and integration using partial fractions. You might also need to use integration methods covered in Further Mathematics, such as integrals leading to inverse trigonometric or inverse hyperbolic functions.

Here is an example in which partial fractions are needed.

Example 2
Find the general solution of the differential equation
$$\frac{dy}{dx} = x(y^2 - y)$$
where $y > 1$.

Solution
Separating the variables gives:
$$\int \frac{1}{y^2 - y}\,dy = \int x\,dx$$
The expression $\frac{1}{y^2 - y}$ can be written as $\frac{1}{y(y-1)}$

This can be written in partial fractions in the form $\frac{A}{y} + \frac{B}{y-1}$

Let $\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$

Multiplying by $y(y-1)$: $\quad 1 = A(y-1) + By$

Note: if you prefer you can find A and B by comparing coefficients

Putting $y = 1 \quad \Rightarrow 1 = B$
Putting $y = 0 \quad \Rightarrow 1 = -A \quad \Rightarrow A = -1$

$$\frac{1}{y(y-1)} = \frac{-1}{y} + \frac{1}{y-1}$$

Returning to the differential equation, we can now write it as:

$$\int \left(\frac{-1}{y} + \frac{1}{y-1} \right) \mathrm{d}y = \int x \, \mathrm{d}x$$

As $y > 1$, modulus signs are not needed

Integrating both sides:

$$-\ln y + \ln(y-1) = \tfrac{1}{2}x^2 + c$$

$$\ln \frac{y-1}{y} = \tfrac{1}{2}x^2 + c$$

Using the laws of logarithms

$$\frac{y-1}{y} = A\mathrm{e}^{x^2/2} \qquad \text{where } A = \mathrm{e}^c$$

$$y - 1 = Ay\mathrm{e}^{x^2/2}$$

$$y(1 - A\mathrm{e}^{x^2/2}) = 1$$

$$y = \frac{1}{1 - A\mathrm{e}^{x^2/2}}$$