Section 2: Mean values and general integration

Notes and examples

These notes contain subsections on
- The mean value of a function
- Summary of integrals

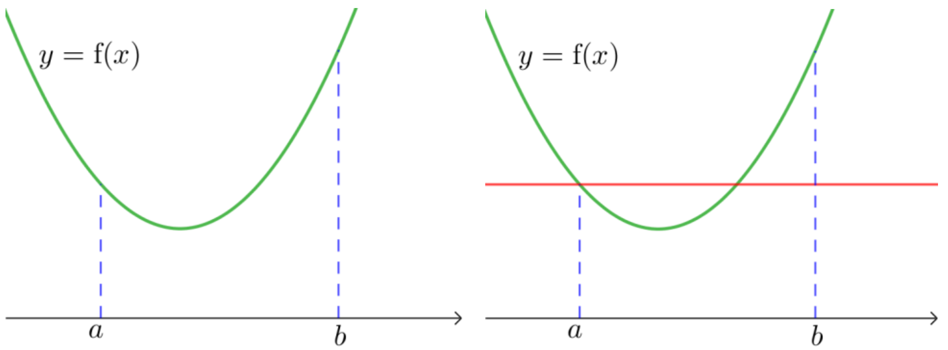
The mean value of a function

Another application of integration is to find the mean value of a function over a specified range of values.

You are familiar with finding the mean of a set of data values: you add up the values and divide by the number of items.

For a function defined over a particular range, there are an infinite number of values. You know that integration is the limit of a sum, so it makes sense that finding the mean value of a function involves integration.

Look at the two diagrams below. The red line on the second diagram has been drawn so that the area under the curve between the two dotted lines is the same as the rectangular area under the red line between the same two dotted lines.



The red line represents the mean of the function.

The area under the curve is given by $\int_a^b f(x)\,dx$.

The area of the rectangle is given by the mean value $\times(b-a)$

So mean value $= \dfrac{1}{b-a}\int_a^b f(x)\,dx$.

Example 1

Find the mean value of the function $y = x^2 - 2x + 3$ between $x = 0$ and $x = 3$.

Solution

$$\text{Mean value } = \frac{1}{3-0}\int_0^3 (x^2-2x+3)\,\mathrm{d}x$$
$$= \tfrac{1}{3}\left[\tfrac{1}{3}x^3 - x^2 + 3x\right]_0^3$$
$$= \tfrac{1}{3}(9-9+9-0)$$
$$= 3$$

Summary of integrals

In your work in A level Further Mathematics, you have now learned to integrate a range of different functions.

You have met the following standard integrals:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right)+c$$
$$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right)+c \quad \text{or } \ln\left(x+\sqrt{x^2+a^2}\right)+c$$
$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right)+c \quad \text{or } \ln\left(x+\sqrt{x^2-a^2}\right)+c \quad (x>a)$$
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$
$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)+c \quad \text{or } \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c$$

These form a family of integrals. More generally, you can integrate any function of the forms $\dfrac{1}{Ax^2+Bx+C}$ or $\dfrac{1}{\sqrt{Ax^2+Bx+C}}$ by completing the square if necessary.

Also, you can often integrate other functions of the form $(a^2-x^2)^{\frac{1}{2}n}$, $(x^2+a^2)^{\frac{1}{2}n}$ or $(x^2-a^2)^{\frac{1}{2}n}$ and by using a trigonometric or hyperbolic substitution. You need to be able to decide which substitution to use; you do this by thinking about trigonometric and hyperbolic identities and choosing one which fits.

So

- for $(a^2-x^2)^{\frac{1}{2}n}$ you could use $x = a\sin u$ as this will give you $(a^2-a^2\sin^2 u)^{\frac{1}{2}n}$ which simplifies to $(a^2\cos^2 u)^{\frac{1}{2}n} = (a\cos u)^n$ (you could alternatively use $x = a\cos u$)
- for $(x^2+a^2)^{\frac{1}{2}n}$ you could use $x = a\sinh u$ as this will give you $(a^2+a^2\sinh^2 u)^{\frac{1}{2}n}$ which simplifies to $(a^2\cosh^2 u)^{\frac{1}{2}n} = (a\cosh u)^n$

- for $(x^2-a^2)^{\frac{1}{2}n}$ you could use $x=a\cosh u$ as this will give you $(a^2\cosh^2 u-a^2)^{\frac{1}{2}n}$ which simplifies to $(a^2\sinh^2 u)^{\frac{1}{2}n}=(a\sinh u)^n$

Once you have used the substitution, you may end up with an integral to do that involves something like $\sin^2 u$ or $\sinh^2 u$. Remember that you can use the double angle formulae, or their hyperbolic equivalents, to integrate functions like these.

Remember that when you do a substitution, you must change back to the original variable at the end, or in the case of a definite integral, it may be easier to change the limits to the new variable instead.

