Section 2: Integrating factors

Exercise level 2

1. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1+x} = x^{\frac{3}{2}} \qquad x \ge 0$$

is to be solved.

(i) Find the particular solution subject to the initial condition $x = 0$, $y = 2$. Describe the behaviour of the solution as x tends to infinity.

A more general case of the above equation is

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + ny = x^{\frac{3}{2}}(1+x)^{2-n} \qquad x \ge 0$$

where n is a real number.

(ii) Solve the differential equation to show that the general solution is

$$y = \frac{2x^{\frac{5}{2}}(7+5x)+C}{35(1+x)^n}$$

where C is an arbitrary constant.

Describe the behaviour of the general solution as x tends to infinity, identifying three distinct cases and the values of n for which they occur.

2. The solution is sought for the differential equation

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4x^3y}{1-x^2} = 1 \qquad (-1 < x < 1).$$

(i) Solve the equation to show that

$$y = \frac{k+3x-x^3}{3(1-x^4)}$$

where k is an arbitrary constant.

(ii) Find the particular solution in each of the cases

(A) $y = 1$ when $x = 0$

(B) $y = 0$ when $x = 0$

In each case, draw a sketch graph of the solution for $-1 < x < 1$.

(iii) Find the value of k for which y tends to a finite limit as x tends to 1.

3. An electrical circuit has total resistance of R ohms and self-inductance L henries and a voltage of V volts is applied when a switch is closed. R, L and V are constants. The current i amperes flowing through it at time t seconds after closing the switch satisfies the equation

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{V}{L}.$$

(i) Given that $i = 0$ when $t = 0$ find the particular solution of the differential equation.

(ii) Sketch the graph of i against t.

(iii) Show that the time taken to reach 95% of its limiting value is independent of the voltage.

Edexcel FM First order DEs 2 Exercise

4. A tank of salt water initially contains 5kg of salt in 1000 l of water. More concentrated salt water is pumped in at a constant rate and the some of the well stirred mixture is pumped out at a slightly higher constant rate.
The amount of salt x kg at time t minutes satisfies the equation

$$\frac{dx}{dt} = -\frac{9}{200-t}x + \frac{2}{5}$$

(i) Find the general solution of this equation.
(ii) Use the initial conditions to find the particular solution.
(iii) The graph below shows the amount of salt in the tank at time t. Explain why the graph is approximately linear for quite large values of t.