Topic Assessment

1. (i) Express $e^{-\frac{1}{2}i\theta} - e^{\frac{1}{2}i\theta}$ in trigonometric form, and show that
$\left(1-e^{i\theta}\right)^2 = -4e^{i\theta}\sin^2\frac{1}{2}\theta$. [6]

(ii) For a positive integer *n*, series C and S are given by

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots + \cos 2n\theta$$

$$S = -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots + \sin 2n\theta$$

Show that $C = (-4)^n \cos n\theta \sin^{2n}\left(\frac{1}{2}\theta\right)$, and find a similar expression for *S*.

[9]

(iii) Given that $w = e^{i\phi}$ is a cube root of 1, state the three possible values of ϕ with $-\pi < \phi < \pi$, and find the possible values of $(1-w)^6$. [5]

2. (a) By considering $(\cos\theta + i\sin\theta)^4$, express $\tan 4\theta$ in terms of $\tan\theta$. [5]

(b) (i) By considering $\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4$, where $z = \cos\theta + i\sin\theta$, show that

$$\sin^2\theta\cos^4\theta = \tfrac{1}{16} + \tfrac{1}{32}\cos 2\theta - \tfrac{1}{16}\cos 4\theta - \tfrac{1}{32}\cos 6\theta.$$ [8]

(ii) Use the substitution $x = \tan\theta$ to show that

$$\int_0^1 \frac{x^2}{\left(1+x^2\right)^4}\,dx = \frac{\pi}{64} + \frac{1}{48}.$$ [7]

3. (a) By considering $(\cos\theta + i\sin\theta)^5$, show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$ [5]

(b) (i) Find the modulus and argument of $-8+8\sqrt{3}i$, and hence find the fourth roots of $-8+8\sqrt{3}i$ in the form $re^{i\theta}$ with $r > 0$ and $-\pi < \theta \le \pi$. [6]

The points representing these fourth roots on an Argand diagram are the vertices of a square. These points are labelled A, B, C, D starting with A in the first quadrant and going anticlockwise.

(ii) Draw an Argand diagram showing the square ABCD.
Find the length of a side of this square. [4]

The midpoints of the sides AB, BC, CD, DA represent complex numbers z_1, z_2, z_3, z_4, which are the fourth roots of a complex number *w*.

(iii) By first finding the modulus and argument of z_1, or otherwise, find *w*, giving your answer in the form *a* + *b*i. [5]

Total 60 marks