Topic assessment

1. (i) Work out the angle between the vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ [3]

   (ii) Show that the lines $\frac{x-2}{-1}=\frac{y-1}{2}=\frac{z}{4}$ and $\frac{x+1}{2}=\frac{y-3}{-5}=\frac{z-2}{3}$ are perpendicular to each other. [2]

2. (i) Find the vector equation of the line joining A(2, -1, 0) to B(3, -2, -5). [3]

   (ii) Verify that A and B both lie on the plane $2x-3y+z=7$. [2]

   (iii) Write down the vector equation of the line passing through A which is perpendicular to the plane. [2]

3. The points A, B and C have coordinates (2, -1, 3), (4, -2, 0) and (1, -5, 0) respectively.

   (i) Work out $\overrightarrow{AB}.\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\overrightarrow{BC}.\begin{pmatrix}1\\-1\\1\end{pmatrix}$. [4]

   (ii) Find the Cartesian equation of the plane ABC. [3]

4. The position vectors of A and B are as follows:

   A: $\mathbf{a}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

   B: $\mathbf{b}=2\mathbf{i}+4\mathbf{j}-\mathbf{k}$

   (i) Find the vector equation of the line AB. [3]

   (ii) The line AB meets the plane $6x-y-3z+13=0$ at the point C. Find the position vector of C. [4]

5. A plane has equation $2x-2y+z=5$.

   (i) Write down a vector normal to the plane. [1]

   (ii) Another plane has equation $px+7z=3$. The angle between the two planes is 60°. Find, in exact form, the possible values of p. [7]

6. Three planes have equations Π_1 $3x-2y-4z=5$

   Π_2 $2x-y=3$

   Π_3 $x+y+12z=1$

   (i) Show that the planes form a triangular prism. [3]

   (ii) Find the angle between the planes Π_1 and Π_2. [4]

7. Four points have coordinates A(3, 9, 5), B(1, 14, 10), C(5, 0, -8) and D(13, -4, 4). The line L passes through A and B, and the line M passes through C and D. Show that the lines L and M intersect, and find the coordinates of their point of intersection. [7]

8. Three points have coordinates A (1, -8, 12), B (-5, 7, 24) and C (3, 5, 8). Find the shortest distance from C to the line AB. [5]

9. Show that the lines $\mathbf{r} = \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 8 \\ -12 \\ -11 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 13 \\ 16 \end{pmatrix}$ are skew, and find the distance between them. [7]

Total 60 marks

Edexcel AS FM Vectors Assessment solutions

Solutions to topic assessment

1. (i) Using the formula: $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$

Let $\underline{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \Rightarrow |\underline{a}| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{4+1+16} = \sqrt{21}$

and $\underline{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \Rightarrow |\underline{b}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

also $\underline{a}.\underline{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 2\times 1 + 1\times(-2) + 4\times 3 = 2 + (-2) + 12 = 12$

So $\cos\theta = \dfrac{12}{\sqrt{14}\sqrt{21}} = 0.699... \Rightarrow \theta = 45.6°$ to 3 s.f. [3]

(ii) The lines are perpendicular if the angle between the two direction vectors is 90°. This means the scalar product of the two direction vectors is 0.

So $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = (-1)\times 2 + 2\times(-5) + 4\times 3 = -2 + (-10) + 12 = 0$

as required. [2]

2. (i) The vector equation of a line is $\underline{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$

So the equation of the line is $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$ [3]

(ii) Substitute the coordinates into the equation of the plane:

A (2, -1, 0): $2\times 2 - 3\times(-1) + 0 = 4 + 3 = 7$

B (3, -2, -5): $2\times 3 - 3\times(-2) + (-5) = 6 + 6 - 5 = 7$ as required. [2]

(iii) A vector normal to the plane is found by the coefficients of x, y and z in the equation of the plane ie. $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

So the equation is: $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. [2]

3. (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$

$\overrightarrow{AB}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2 + 1 - 3 = 0$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$

$\overrightarrow{BC}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -3 + 3 + 0 = 0$

[4]

(ii) The vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is perpendicular to two lines on the plane and so it must be perpendicular to the plane.
The equation of the plane is $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$

$$\underset{\sim}{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2 + 1 + 3 = 6$$

So the Cartesian equation of the plane is $x - y + z = 6$ [3]

Edexcel AS FM Vectors Assessment solutions

4. (i) Direction vector is $\underline{b}-\underline{a}=\begin{pmatrix}2\\4\\-1\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}1\\6\\-4\end{pmatrix}$

The equation of the line is $\underline{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\6\\-4\end{pmatrix}$. [3]

(ii) The line AB is: $\underline{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\6\\-4\end{pmatrix}$

So reading across: $x=1+\lambda$

$y=-2+6\lambda$

$z=3-4\lambda$

Substituting into the equation of the plane $6x-y-3z+13=0$:

$6(1+\lambda)-(-2+6\lambda)-3(3-4\lambda)+13=0$

$12+12\lambda=0$

$\Rightarrow \lambda=-1$

Substitute $\lambda=-1$ into the line AB: $\underline{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}-1\begin{pmatrix}1\\6\\-4\end{pmatrix}=\begin{pmatrix}0\\-8\\7\end{pmatrix}$

So the position vector of C is: $\underline{c}=-8\mathbf{j}+7\mathbf{k}$ [4]

5. (i) Normal vector is $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ [1]

(ii) Normal vector for second plane is $\underset{\sim}{n}_2=\begin{pmatrix}p\\0\\7\end{pmatrix} \Rightarrow |\underset{\sim}{n}_2|^2=p^2+49$

For first plane, $\underset{\sim}{n}_1=\begin{pmatrix}2\\-2\\1\end{pmatrix} \Rightarrow |\underset{\sim}{n}_1|^2=4+4+1=9$

$\underset{\sim}{n}_1.\underset{\sim}{n}_2=2p+7$

$\cos\theta = \dfrac{\underset{\sim}{n}_1 . \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|}$

$\dfrac{1}{2} = \dfrac{2p+7}{3\sqrt{p^2+49}}$

$3\sqrt{p^2+49} = 2(2p+7)$

Squaring both sides: $9(p^2+49) = 4(2p+7)^2$

$9p^2 + 441 = 16p^2 + 112p + 196$

$7p^2 + 112p - 245 = 0$

$p^2 + 16p - 35 = 0$

$p = \dfrac{-16 \pm \sqrt{16^2 - 4\times 1\times -35}}{2} = \dfrac{-16 \pm \sqrt{396}}{2} = -8 \pm 3\sqrt{11}$

[7]

6. (i) (1) $3x - 2y - 4z = 5$

(2) $2x - y = 3$

(3) $x + y + 12z = 1$

$(1) + 2\times(3) \Rightarrow 5x + 20z = 7 \Rightarrow x + 4z = \frac{7}{5}$

$(2) + (3) \Rightarrow 3x + 12z = 4 \Rightarrow x + 4z = \frac{4}{3}$

So the equations are inconsistent, and since no equation is a multiple of any other, none of the planes are parallel. So they must form a triangular prism.

[3]

(ii) Angle between planes is angle between the normal vectors $\underset{\sim}{n}_1 = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$ and

$\underset{\sim}{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$\underset{\sim}{n}_1 . \underset{\sim}{n}_2 = 6 + 2 + 0 = 8$

$|\underset{\sim}{n}_1| = \sqrt{9+4+16} = \sqrt{29}$

$|\underset{\sim}{n}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\cos\theta = \dfrac{8}{\sqrt{29}\sqrt{5}}$

$\theta = 48.4^\circ$

The angle between the planes is 48.4° (3 s.f.) [4]

7. The equation of AB is $\underline{r} = \begin{pmatrix} 3 \\ 9 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 5 \end{pmatrix}$.

The equation of CD is $\underline{r} = \begin{pmatrix} 5 \\ 0 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

At the point of intersection, if there is one,

$$3 - 2\lambda = 5 + 2\mu \qquad \lambda + \mu = -1 \quad ①$$
$$9 + 5\lambda = -\mu \quad \Rightarrow \quad 5\lambda + \mu = -9 \quad ②$$
$$5 + 5\lambda = -8 + 3\mu \qquad 5\lambda - 3\mu = -13 \quad ③$$

② - ① $\Rightarrow 4\lambda = -8$

$\Rightarrow \lambda = -2,\ \mu = 1$

In equation ③, LHS = $5 \times -2 - 3 \times 1 = -13$ = RHS.
The three equations are consistent, so the lines intersect.
The point of intersection is (7, -1, -5).

[7]

8. $\overrightarrow{AB} = \begin{pmatrix} -6 \\ 15 \\ 12 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$

Equation of AB is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -8 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$

Let M be the point on AB nearest to C

$$\overrightarrow{OM} = \begin{pmatrix} 1 - 2\lambda \\ -8 + 5\lambda \\ 12 + 4\lambda \end{pmatrix}$$

$$\overrightarrow{CM} = \begin{pmatrix} 1 - 2\lambda \\ -8 + 5\lambda \\ 12 + 4\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} -2 - 2\lambda \\ -13 + 5\lambda \\ 4 + 4\lambda \end{pmatrix}$$

$\overrightarrow{CM}$ is perpendicular to AB, so $\begin{pmatrix} -2-2\lambda \\ -13+5\lambda \\ 4+4\lambda \end{pmatrix} . \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} = 0$

$$-2(-2-2\lambda)+5(-13+5\lambda)+4(4+4\lambda)=0$$
$$4+4\lambda-65+25\lambda+16+16\lambda=0$$
$$45\lambda=45$$
$$\lambda=1$$

$$\overrightarrow{CM} = \begin{pmatrix} -4 \\ -8 \\ 8 \end{pmatrix} = 4\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{CM} = 4\sqrt{1^2+2^2+2^2} = 4\sqrt{9} = 12$$

[5]

9. If the lines intersect, $\begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -12 \\ -11 \end{pmatrix} + \mu\begin{pmatrix} -5 \\ 13 \\ 16 \end{pmatrix}$

so $2+2\lambda = 8-5\mu$

$8+2\lambda = -12+13\mu$

$-1-\lambda = -11+16\mu$

(1) $2\lambda+5\mu=6$

(2) $2\lambda-13\mu=-20$

(3) $\lambda+16\mu=10$

(1) – (2) $\Rightarrow 18\mu = 26 \quad \Rightarrow \mu = \frac{13}{9}, \lambda = -\frac{11}{18}$

Substituting into (3): $\lambda+16\mu = -\frac{11}{18}+16\times\frac{13}{9} = \frac{45}{2}$

so there are no values of **λ** and **μ** which satisfy all three equations, and so the lines do not intersect.

The direction vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 13 \\ 16 \end{pmatrix}$ are not scalar multiples of each other, so they are not parallel and therefore the lines are not parallel.

Therefore the lines are skew.

Let P and Q be the points on the two lines that are closest to each other.

$$\overrightarrow{OP} = \begin{pmatrix} 2+2\lambda \\ 8+2\lambda \\ -1-\lambda \end{pmatrix} \text{ and } \overrightarrow{OQ} = \begin{pmatrix} 8-5\mu \\ -12+13\mu \\ -11+16\mu \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 8-5\mu \\ -12+13\mu \\ -11+16\mu \end{pmatrix} - \begin{pmatrix} 2+2\lambda \\ 8+2\lambda \\ -1-\lambda \end{pmatrix} = \begin{pmatrix} 6-5\mu-2\lambda \\ -20+13\mu-2\lambda \\ -10+16\mu+\lambda \end{pmatrix}$$

$\overrightarrow{PQ}$ is perpendicular to $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

so $\begin{pmatrix} 6-5\mu-2\lambda \\ -20+13\mu-2\lambda \\ -10+16\mu+\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$

$$2(6-5\mu-2\lambda)+2(-20+13\mu-2\lambda)-(-10+16\mu+\lambda)=0$$
$$-18-9\lambda=0$$
$$\lambda=-2$$

$\overrightarrow{PQ}$ is perpendicular to $\underset{\sim}{r} = \begin{pmatrix} 8 \\ -12 \\ -11 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 13 \\ 16 \end{pmatrix}$

so $\begin{pmatrix} 6-5\mu-2\lambda \\ -20+13\mu-2\lambda \\ -10+16\mu+\lambda \end{pmatrix} . \begin{pmatrix} -5 \\ 13 \\ 16 \end{pmatrix} = 0$

$$-5(6-5\mu-2\lambda)+13(-20+13\mu-2\lambda)+16(-10+16\mu+\lambda)=0$$
$$-450+450\mu=0$$
$$\mu=1$$

$$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{5^2+3^2+4^2} = \sqrt{50} = 5\sqrt{2}$$

So the distance between the lines is $5\sqrt{2}$.

[7]