Topic assessment

1. Find $\sum_{r=1}^{n}(6r-1)(4r-1)$, giving your answer in its simplest form. [5]

2. Prove by induction that
$$\sum_{r=1}^{n}(3r^2-2r)=\tfrac{1}{2}n(n+1)(2n-1)$$ [7]

3. Prove by induction that
$$\begin{pmatrix}-1 & -4\\ 1 & 3\end{pmatrix}^n=\begin{pmatrix}1-2n & -4n\\ n & 2n+1\end{pmatrix}.$$
[6]

4. A sequence of real numbers $u_1, u_2, u_3, \ldots$ is defined by
$$u_1=-1 \quad \text{and} \quad u_{n+1}=\frac{3u_n}{5u_n+6} \quad \text{for } n\geq 1.$$
Prove by induction that $u_n=\dfrac{3}{2^n-5}$. [6]

5. Find the sum of the series
$$1\times 9 + 2\times 33 + 3\times 73 + \ldots + n(8n^2+1),$$
giving your answer in a fully factorised form. [5]

6. Prove by induction that 13^n-6^{n-2} is a multiple of 7 for all $n\geq 2$.
[7]

7. Prove by induction that $\sum_{r=1}^{n} r3^r=\frac{3}{4}+\frac{1}{4}(2n-1)3^{n+1}$. [7]

8. Prove by induction that $\sum_{r=1}^{n}\dfrac{2^r(r^2-2)}{(r+1)^2(r+2)^2}=\dfrac{2^{n+1}}{(n+2)^2}-\dfrac{1}{2}$ [7]

Total 50 marks

Solutions to topic assessment

1. $$\sum_{r=1}^{n}(6r-1)(4r-1)=\sum_{r=1}^{n}(24r^2-10r+1)$$
$$=24\sum_{r=1}^{n}r^2-10\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$
$$=24\times\tfrac{1}{6}n(n+1)(2n+1)-10\times\tfrac{1}{2}n(n+1)+n$$
$$=n(4(n+1)(2n+1)-5(n+1)+1)$$
$$=n\left(8n^2+12n+4-5n-5+1\right)$$
$$=n\left(8n^2+7n\right)$$
$$=n^2(8n+7)$$

[5]

2. To prove: $\sum_{r=1}^{n}(3r^2-2r)=\tfrac{1}{2}n(n+1)(2n-1)$.

When $n=1$, L.H.S. $=3-2=1$

R.H.S. $=\tfrac{1}{2}\times1\times2\times1=1$

So the result is true for $n=1$.

Assume $\sum_{r=1}^{k}(3r^2-2r)=\tfrac{1}{2}k(k+1)(2k-1)$

$$\sum_{r=1}^{k+1}(3r^2-2r)=\tfrac{1}{2}k(k+1)(2k-1)+3(k+1)^2-2(k+1)$$
$$=\tfrac{1}{2}(k+1)\left(2k^2-k+6k+6-4\right)$$
$$=\tfrac{1}{2}(k+1)(2k^2+5k+2)$$
$$=\tfrac{1}{2}(k+1)(k+2)(2k+1)$$
$$=\tfrac{1}{2}(k+1)(k+2)(2(k+1)-1)$$

So if the result is true for $n=k$, then it is true for $n=k+1$.
Since it is true for $n=1$, then it is true for all positive integers greater than or equal to 1 by induction.

[7]

3. To prove: $\begin{pmatrix}-1 & -4\\ 1 & 3\end{pmatrix}^n=\begin{pmatrix}1-2n & -4n\\ n & 2n+1\end{pmatrix}$

For $n=1$, $\begin{pmatrix}1-2n & -4n\\ n & 2n+1\end{pmatrix}=\begin{pmatrix}1-2 & -4\\ 1 & 2+1\end{pmatrix}=\begin{pmatrix}-1 & -4\\ 1 & 3\end{pmatrix}$

so true for $n=1$.

Assume true for $n = k$, so $\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}^k = \begin{pmatrix} 1-2k & -4k \\ k & 2k+1 \end{pmatrix}$

For $n = k+1$,

$$\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1-2k & -4k \\ k & 2k+1 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2k-4k & -4+8k-12k \\ -k+2k+1 & -4k+6k+3 \end{pmatrix}$$

$$= \begin{pmatrix} -1-2k & -4-4k \\ k+1 & 2k+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2(k+1) & -4(k+1) \\ k+1 & 2(k+1)+1 \end{pmatrix}$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

[6]

4. To prove that $u_n = \dfrac{3}{2^n - 5}$.

When $n = 1$, $\quad u_1 = \dfrac{3}{2-5} = \dfrac{3}{-3} = -1$

So the result is true for $n = 1$.

Assume $u_k = \dfrac{3}{2^k - 5}$

$$u_{k+1} = \frac{3u_k}{5u_k + 6}$$

$$= \frac{3\left(\frac{3}{2^k-5}\right)}{5\left(\frac{3}{2^k-5}\right)+6}$$

$$= \frac{9}{15+6(2^k-5)}$$

$$= \frac{3}{5+2\times 2^k - 10}$$

$$= \frac{3}{2^{k+1}-5}$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

[6]

5. $1\times 9 + 2\times 33 + 3\times 73 + \ldots + n(8n^2+1) = \sum_{r=1}^{n} r(8r^2+1)$

$$= 8\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r$$
$$= 8\times \tfrac{1}{4}n^2(n+1)^2 + \tfrac{1}{2}n(n+1)$$
$$= \tfrac{1}{2}n(n+1)\left(4n(n+1)+1\right)$$
$$= \tfrac{1}{2}n(n+1)\,(4n^2+4n+1)$$
$$= \tfrac{1}{2}n(n+1)\,(2n+1)^2$$

[5]

6. To prove that $13^n - 6^{n-2}$ is a multiple of 7 for all $n \geq 2$.

For $n = 2$, $13^n - 6^{n-2} = 13^2 - 6^0 = 169 - 1 = 168$
so it is true for $n = 2$.

Assume it is true for $n = k$, so $13^k - 6^{k-2} = 7p$ for some integer p.
For $n = k+1$, $13^{k+1} - 6^{(k+1)-2} = 13\times 13^k - 6^{k-1}$

$$= 13(7p + 6^{k-2}) - 6^{k-1}$$
$$= 91p + 13\times 6^{k-2} - 6^{k-1}$$
$$= 91p + 13\times 6^{k-2} - 6\times 6^{k-2}$$
$$= 91p + 7\times 6^{k-2}$$
$$= 7(13p - 6^{k-2})$$

So it is a multiple of 7 for $n = k+1$.

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

[7]

7. To prove that $\sum_{r=1}^{n} r3^r = \tfrac{3}{4} + \tfrac{1}{4}(2n-1)3^{n+1}$.

When $n = 1$, L.H.S. $= 1\times 3 = 3$

R.H.S. $= \tfrac{3}{4} + \tfrac{1}{4}\times 1\times 3^2 = \tfrac{3}{4} + \tfrac{9}{4} = \tfrac{12}{4} = 3$

So the result is true for $n = 1$.

Assume $\sum_{r=1}^{k} r3^r = \tfrac{3}{4} + \tfrac{1}{4}(2k-1)3^{k+1}$

$$\sum_{r=1}^{k+1} r3^r = \tfrac{3}{4} + \tfrac{1}{4}(2k-1)3^{k+1} + (k+1)3^{k+1}$$
$$= \tfrac{3}{4} + \tfrac{1}{4}\left(2k-1+4(k+1)\right)3^{k+1}$$
$$= \tfrac{3}{4} + \tfrac{1}{4}(6k+3)3^{k+1}$$
$$= \tfrac{3}{4} + \tfrac{1}{4}(2k+1)\times 3\times 3^{k+1}$$
$$= \tfrac{3}{4} + \tfrac{1}{4}\left(2(k+1)-1\right)3^{k+2}$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

[7]

8. To prove that $\sum_{r=1}^{n} \frac{2^r(r^2-2)}{(r+1)^2(r+2)^2} = \frac{2^{n+1}}{(n+2)^2} - \frac{1}{2}$.

When $n = 1$, L.H.S. $= \frac{2(1-2)}{2^2 \times 3^2} = -\frac{1}{18}$

R.H.S. $= \frac{2^2}{3^2} - \frac{1}{2} = \frac{4}{9} - \frac{1}{2} = -\frac{1}{18}$

So the result is true for $n = 1$.

Assume $\sum_{r=1}^{k} \frac{2^r(r^2-2)}{(r+1)^2(r+2)^2} = \frac{2^{k+1}}{(k+2)^2} - \frac{1}{2}$

$$\sum_{r=1}^{k+1} \frac{2^r(r^2-2)}{(r+1)^2(r+2)^2} = \frac{2^{k+1}}{(k+2)^2} - \frac{1}{2} + \frac{2^{k+1}((k+1)^2-2)}{(k+2)^2(k+3)^2}$$
$$= \frac{2^{k+1}(k+3)^2 + 2^{k+1}((k+1)^2-2)}{(k+2)^2(k+3)^2} - \frac{1}{2}$$
$$= \frac{2^{k+1}(k^2+6k+9+k^2+2k+1-2)}{(k+2)^2(k+3)^2} - \frac{1}{2}$$
$$= \frac{2^{k+1}(2k^2+8k+8)}{(k+2)^2(k+3)^2} - \frac{1}{2}$$
$$= \frac{2^{k+1}\times 2(k+2)^2}{(k+2)^2(k+3)^2} - \frac{1}{2}$$
$$= \frac{2^{k+2}}{(k+3)^2} - \frac{1}{2}$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

[7]