Topic assessment

1. The quadratic equation $2z^2 - 4z + 5 = 0$ has roots α and β.
   (i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]
   (ii) Find the quadratic equation with roots $3\alpha - 1$, $3\beta - 1$. [3]
   (iii) Find the cubic equation which has roots α, β and $\alpha + \beta$. [4]

2. The equation $z^3 + kz^2 - 4z - 12 = 0$ has roots α, β and γ.
   (i) Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and express k in terms of α, β and γ. [3]
   (ii) For the case where $\gamma = -\alpha$, solve the equation and find the value of k. [4]
   (iii) For the case $k = 5$, find a cubic equation with roots $2 - \alpha$, $2 - \beta$, $2 - \gamma$. [4]

3. The complex number 3 + 2i is a root of the equation $2x^3 + px^2 + 20x + q = 0$, where p and q are real numbers.
   (i) Find the other two roots of the cubic equation. [4]
   (ii) Find the values of p and q. [4]

4. The complex number 2 + 3i is a root of the equation
   $$4z^4 - 12z^3 + 33z^2 + 64z - 39 = 0$$
   Solve the equation. [5]

5. The cubic equation $2z^3 + pz^2 + qz + r = 0$ has roots $\frac{\alpha}{k}, \alpha, k\alpha$.
   (i) Express p, q and r in terms of k and α. [3]
   (ii) Show that $2q^3 = p^3 r$. [3]
   (iii) Solve the equation for the case where $p = q = -3$. [4]

6. The equation $x^4 - 6x^3 - 73x^2 + kx + m = 0$ has two positive roots, α, β and two negative roots γ, δ. It is given that $\alpha\beta = \gamma\delta = 4$.
   (i) Find the values of the constants k and m. [4]
   (ii) Show that $(\alpha + \beta)(\gamma + \delta) = -81$. [3]
   (iii) Find the quadratic equation which has roots $\alpha + \beta$ and $\gamma + \delta$. [2]
   (iv) Find $\alpha + \beta$ and $\gamma + \delta$. [3]
   (v) Show that $\alpha^2 - 3(1 + \sqrt{10})\alpha + 4 = 0$, and find similar quadratic equations satisfied by β, γ and δ. [5]

Total 60 marks

Solutions to topic assessment

1. (i) $2z^2 - 4z + 5 = 0$

$\alpha + \beta = -\frac{b}{a} = -\frac{-4}{2} = 2$

$\alpha\beta = \frac{c}{a} = \frac{5}{2}$

[2]

(ii) Let $w = 3z - 1$, so $z = \frac{w+1}{3}$

Substituting into quadratic equation:

$$2\left(\frac{w+1}{3}\right)^2 - 4\left(\frac{w+1}{3}\right) + 5 = 0$$

$$2(w+1)^2 - 12(w+1) + 45 = 0$$

$$2w^2 + 4w + 2 - 12w - 12 + 45 = 0$$

$$2w^2 - 8w + 35 = 0$$

[3]

(iii) The original quadratic equation has roots α and β.

The value of $\alpha + \beta = 2$

The cubic equation is therefore $(2z^2 - 4z + 5)(z - 2) = 0$

$$2z^3 - 4z^2 + 5z - 4z^2 + 8z - 10 = 0$$

$$2z^3 - 8z^2 + 13z - 10 = 0$$

[4]

2. (i) $z^3 + kz^2 - 4z - 12 = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -4$

$\alpha\beta\gamma = -\frac{d}{a} = 12$

$\alpha + \beta + \gamma = -\frac{b}{a} = -k$

$k = -\alpha - \beta - \gamma$

[3]

(ii) If $\gamma = -\alpha$: $\alpha\beta - \beta\alpha - \alpha^2 = -4$

$\alpha^2 = 4$

$\alpha = \pm 2$

$-\alpha^2\beta = 12$

$-4\beta = 12$

$\beta = -3$

The roots are 2, -2 and -3

$k = -2 + 2 + 3 = 3$

[4]

(iii) $z^3 + 5z^2 - 4z - 12 = 0$

$w = 2 - z$ so $z = 2 - w$

Substituting into cubic equation:

$$(2-w)^3 + 5(2-w)^2 - 4(2-w) - 12 = 0$$

$$8 - 12w + 6w^2 - w^3 + 20 - 20w + 5w^2 - 8 + 4w - 12 = 0$$

$$w^3 - 11w^2 + 28w - 8 = 0$$

[4]

3. (i) 3 + 2i is a root, so 3 – 2i is also a root.

$$\sum \alpha\beta = \frac{c}{a}$$

$$(3+2i)\alpha + (3-2i)\alpha + (3+2i)(3-2i) = \frac{20}{2}$$

$$6\alpha + 13 = 10$$

$$6\alpha = -3$$

$$\alpha = -\tfrac{1}{2}$$

The other two roots are 3 – 2i and $-\tfrac{1}{2}$.

[4]

(ii) $\sum \alpha = -\frac{b}{a}$

$$3 + 2i + 3 - 2i - \tfrac{1}{2} = -\frac{p}{2}$$

$$5.5 = -\frac{p}{2}$$

$$p = -11$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$-\tfrac{1}{2}(3+2i)(3-2i) = -\frac{q}{2}$$

$$q = 13$$

[4]

4. $2 + 3i$ is a root so $2 - 3i$ is also a root

For the quadratic with roots $\alpha = 2 + 3i$ and $\beta = 2 - 3i$,

$\alpha + \beta = 4$ and $\alpha\beta = 13$

so the quadratic factor is $z^2 - 4z + 13$

$$4z^4 - 12z^3 + 33z^2 + 64z - 39 = 0$$

$$(z^2 - 4z + 13)(4z^2 + 4z - 3) = 0$$

$$(z^2 - 4z + 13)(2z + 3)(2x - 1) = 0$$

The roots are $2+3i$, $2-3i$, $-\frac{3}{2}$ and $\frac{1}{2}$

[5]

5. (i) $2z^3 + pz^2 + qz + r = 0$

$$\sum \alpha = -\frac{b}{a}$$

$$\frac{\alpha}{k} + \alpha + k\alpha = -\frac{p}{2}$$

$$p = -2\alpha\left(\frac{1}{k} + 1 + k\right)$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\left(\frac{\alpha}{k} \times \alpha\right) + (\alpha \times k\alpha) + \left(k\alpha \times \frac{\alpha}{k}\right) = \frac{q}{2}$$

$$q = 2\alpha^2\left(\frac{1}{k} + k + 1\right)$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$\frac{\alpha}{k} \times \alpha \times k\alpha = -\frac{r}{2}$$

$$r = -2\alpha^3$$

[3]

(ii) Dividing the first two equations: $\frac{p}{q} = \frac{-2\alpha}{2\alpha^2}$

$$\alpha = -\frac{q}{p}$$

Substituting into third equation: $r = -2\left(-\frac{q}{p}\right)^3$

$$p^3 r = 2q^3$$

[3]

(iii) $p = q = -3 \quad \Rightarrow r = 2$

$r = -2\alpha^3 \quad \Rightarrow 2 = -2\alpha^3 \quad \Rightarrow \alpha = -1$

$$p = -2\alpha\left(\frac{1}{k} + 1 + k\right)$$

$$-3 = 2\left(\frac{1}{k} + 1 + k\right)$$

$$-3k = 2 + 2k + 2k^2$$

$$2k^2 + 5k + 2 = 0$$

$$(2k+1)(k+2) = 0$$

$$k = -\tfrac{1}{2} \quad \text{or} \quad -2$$

The roots are $\frac{1}{2}$, -1 and 2.

[4]

6. (i) $x^4 - 6x^3 - 73x^2 + kx + m = 0$

$$\alpha\beta\gamma\delta = m$$

$$4 \times 4 = m$$

$$m = 16$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -k$$

$$4\gamma + 4\beta + 4\alpha + 4\delta = -k$$

$$k = -4(\alpha + \beta + \gamma + \delta)$$

$$k = -4 \times -\frac{b}{a}$$

$$k = 4 \times -6 = -24$$

[4]

(ii) $(\alpha+\beta)(\gamma+\delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta$

$$= \sum \alpha\beta - (\alpha\beta + \gamma\delta)$$

$$= \frac{c}{a} - (4+4)$$

$$= -73 - 8$$

$$= -81$$

[3]

(iii) Sum of roots of quadratic equation $= \alpha + \beta + \gamma + \delta = 6$

Product of roots of quadratic equation $= (\alpha+\beta)(\gamma+\delta) = -81$

Quadratic equation is therefore $x^2 - 6x - 81 = 0$

[2]

(iv) $x^2 - 6x - 81 = 0$

$$x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times -81}}{2} = 3 \pm 3\sqrt{10}$$

$$\alpha + \beta = 3 + 3\sqrt{10}$$

$$\gamma + \delta = 3 - 3\sqrt{10}$$

[3]

(v) $\alpha + \beta = 3 + 3\sqrt{10}$ and $\alpha\beta = 4$

so α and β are the roots of the quadratic equation

$$x^2 - 3(1 + \sqrt{10})x + 4 = 0$$

so α satisfies $\alpha^2 - 3(1 + \sqrt{10})\alpha + 4 = 0$

and β satisfies $\beta^2 - 3(1 + \sqrt{10})\beta + 4 = 0$

$\gamma + \delta = 3 - 3\sqrt{10}$ and $\gamma\delta = 4$

so γ and δ are the roots of the quadratic equation

$$x^2 - 3(1 - \sqrt{10})x + 4 = 0$$

so γ satisfies $\gamma^2 - 3(1 - \sqrt{10})\gamma + 4 = 0$

and δ satisfies $\delta^2 - 3(1 - \sqrt{10})\delta + 4 = 0$

[5]