Edexcel AS Further Maths Complex numbers

Topic Assessment

1. (a) Solve the equation $z^2 + 2z + 10 = 0$.
Find the modulus and argument of each root. [4]

(b) Complex numbers α and β are given by

$$\alpha = 2\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right), \qquad \beta = 4\sqrt{2}\left(\cos\frac{5\pi}{8} + \mathrm{i}\sin\frac{5\pi}{8}\right).$$

(i) Write down the modulus and argument of each of the complex numbers α and β.
Illustrate these two complex numbers on an Argand diagram. [3]

(ii) Indicate a length on your diagram which is equal to $|\beta - \alpha|$, and show that $|\beta - \alpha| = 6$. [3]

(iii) On your diagram, draw and label

(A) the locus L of points representing complex numbers z such that $|z - \alpha| = 6$, [3]

(B) the locus M of points representing complex numbers z such that $\arg(z - \alpha) = \frac{5\pi}{8}$. [3]

2. The cubic equation $z^3 + z^2 + 4z - 48 = 0$ has one real root α and two complex roots β and γ.

(i) Verify that $\alpha = 3$ and find β and γ in the form $a + b\mathrm{i}$. Take β to be the root with positive imaginary part, and give your answers in an exact form. [5]

(ii) Find the modulus and argument of each of the numbers α, β, γ, giving the arguments in radians between $-\pi$ and π.
Illustrate the three numbers on an Argand diagram. [5]

(iii) On your Argand diagram, draw the locus of points representing complex numbers z such that

$$\arg(z - \alpha) = \arg\beta.$$ [2]

3. You are given that the complex number $\alpha = 1 + 4\mathrm{i}$ satisfies the cubic equation

$$z^3 + 5z^2 + kz + m = 0,$$

where k and m are real constants.

(i) Find α^2 and α^3 in the form $a + b\mathrm{i}$. [3]

(ii) Find the value of k, and show that $m = 119$. [4]

(iii) Find the other two roots of the cubic equation. Give the arguments of all three roots. [6]

(iv) Verify that there is a constant c such that all three roots of the cubic equation satisfy

$$|z + 2| = c.$$

Draw an Argand diagram showing the locus of points representing all complex numbers z for which $|z + 2| = c$. Mark the points corresponding to the three roots of the cubic equation. [6]

4. (i) Describe in words the locus L_1 of points representing complex numbers w which satisfy

$$|w-9\text{i}|=|w-12|.$$

Draw a diagram showing L_1. [5]

The locus L_2 consists of the points representing complex numbers z for which $|z-9\text{i}|=2|z-12|$.

(ii) By writing $z = x + y\text{i}$, where x and y are real, obtain an equation relating x and y, and hence show that L_2 is a circle. Give the centre and radius of this circle. [6]

(iii) Hence write down an equation for L_2 in which z occurs only once. [2]

Total 60 marks

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Solutions to Topic Assessment

1. (a) $z^2 + 2z + 10 = 0$

Using the quadratic formula, $z = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 10}}{2}$

$= \dfrac{-2 \pm \sqrt{-36}}{2}$

$= \dfrac{-2 \pm 6i}{2}$

$= -1 \pm 3i$

$|-1 + 3i| = \sqrt{1^2 + 3^2} = \sqrt{10}$

(-1 + 3i) is in the second quadrant,

so $\arg(-1 + 3i) = \arctan\left(\dfrac{3}{-1}\right) + \pi = 1.89$ (3 s.f.)

$|-1 - 3i| = \sqrt{10}$

$\arg(-1 - 3i) = -1.89$

[4]

(b) (i) $\alpha = 2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)$

$|\alpha| = 2$

$\arg\alpha = \dfrac{\pi}{8}$

$\beta = 4\sqrt{2}\left(\cos\dfrac{5\pi}{8} + i\sin\dfrac{5\pi}{8}\right)$

$|\beta| = 4\sqrt{2}$

$\arg\beta = \dfrac{5\pi}{8}$

[3]

(ii) The triangle is a right-angled triangle,

so $|\beta - \alpha|^2 = 2^2 + (4\sqrt{2})^2$

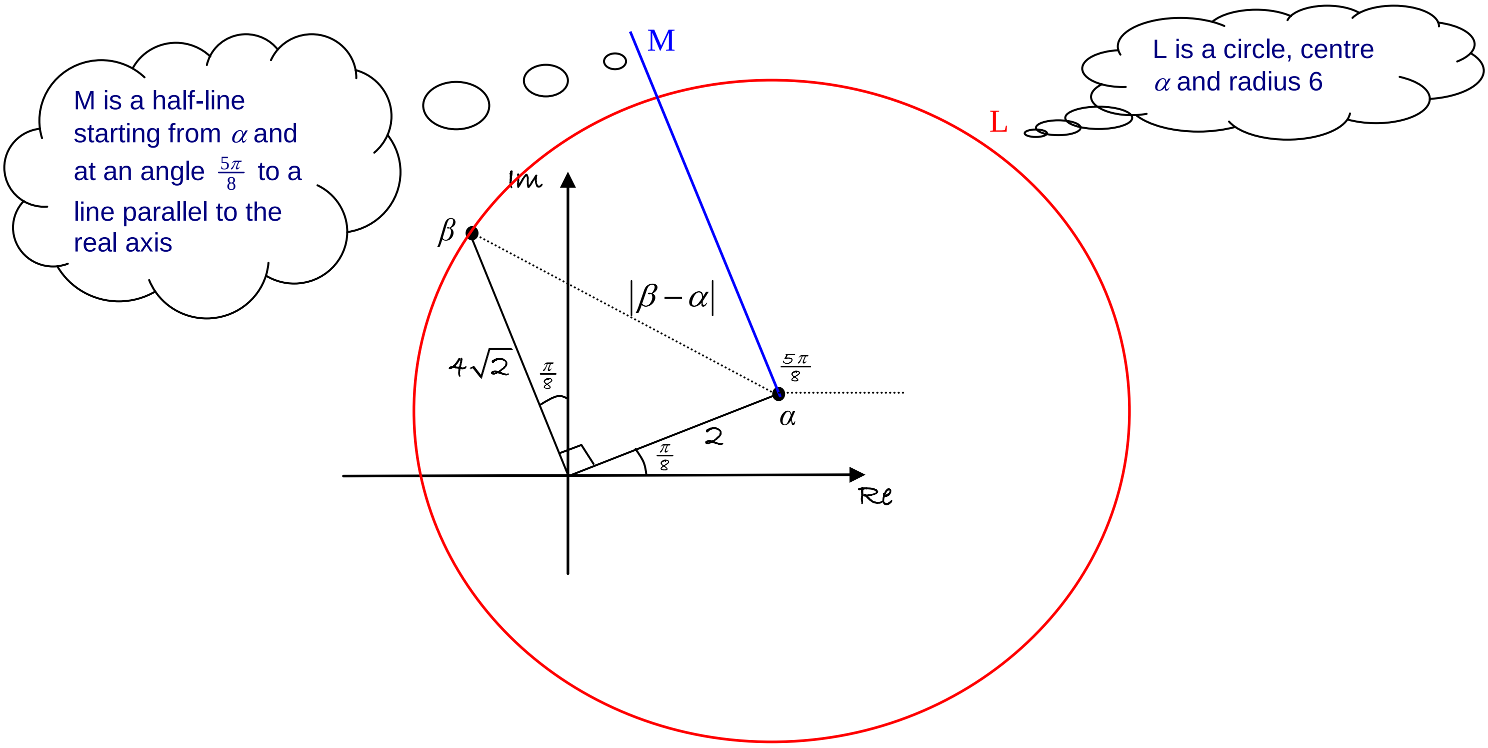
$= 4 + 32$

$= 36$

$|\beta - \alpha| = 6$

[3]

(iii)



[3]

2. (i) $f(z)=z^3+z^2+4z-48$

$f(3)=3^3+3^2+4\times3-48$

$=27+9+12-48$

$=0$

3 is a root, so the real root $\alpha=3$.

$z^3+z^2+4z-48=0$

$(z-3)(z^2+4z+16)=0$

The complex roots are the roots of the quadratic equation

$z^2+4z+16=0$.

$$z=\frac{-4\pm\sqrt{4^2-4\times1\times16}}{2}$$

$$=\frac{-4\pm\sqrt{-48}}{2}$$

$$=\frac{-4\pm4i\sqrt{3}}{2}$$

$$=-2\pm2i\sqrt{3}$$

The complex roots are $\beta=-2+2i\sqrt{3}$, $\gamma=-2-2i\sqrt{3}$.

[5]

(ii) $|\alpha|=3$

$\arg\alpha=0$

$|\beta| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

β is in the second quadrant, so $\arg\beta = \pi + \arctan\left(\frac{2\sqrt{3}}{-2}\right)$

$= \pi + \arctan\left(-\sqrt{3}\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

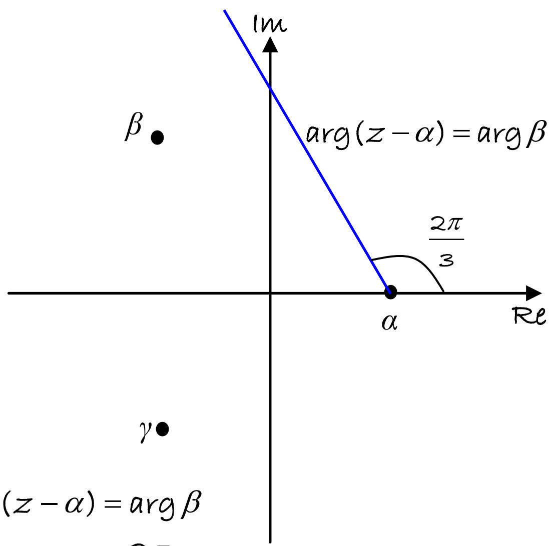
$|\gamma| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

γ is in the third quadrant, so $\arg\gamma = \arctan\left(\frac{-2\sqrt{3}}{-2}\right) - \pi$

$= \arctan\left(\sqrt{3}\right) - \pi$

$= \frac{\pi}{3} - \pi$

$= -\frac{2\pi}{3}$



[5]

(iii) $\arg(z-\alpha) = \arg\beta$

$\arg(z-3) = \frac{2\pi}{3}$

The locus is the half-line starting from $z = 3$ in the direction $\frac{2\pi}{3}$ (shown on Argand diagram above).

[2]

3. (i) $\alpha = 1 + 4i$

$\alpha^2 = (1+4i)(1+4i) = 1 + 8i - 16 = -15 + 8i$

$\alpha^3 = (-15+8i)(1+4i) = -15 - 52i - 32 = -47 - 52i$

[3]

(ii) $\alpha^3 + 5\alpha^2 + k\alpha + m = 0$

$-47 - 52i + 5(-15 + 8i) + k(1 + 4i) + m = 0$

Equating imaginary parts: $-52 + 40 + 4k = 0$

$\Rightarrow k = 3$

Equating real parts: $-47 - 75 + k + m = 0$

$\Rightarrow m = 122 - k = 119$

[4]

(iii) $\alpha = 1 + 4i$ is a root, so $\alpha^* = 1 - 4i$ is another root.

A quadratic factor is $(z - 1 - 4i)(z - 1 + 4i) = (z-1)^2 + 16$

$= z^2 - 2z + 1 + 16$

$= z^2 - 2z + 17$

$z^3 + 5z^2 + 3z + 119 = 0$

$(z^2 - 2z + 17)(z + 7) = 0$

The third root is z = -7.

$\arg\alpha = \arctan\left(\frac{4}{1}\right) = 1.33$ (3 s.f.)

By symmetry $\arg\alpha^* = -1.33$ (3 s.f.)

$\arg(-7) = \pi$.

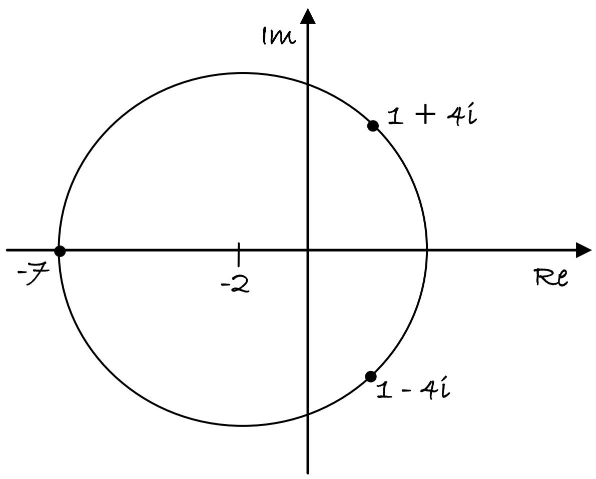
[6]

(iv) $|\alpha + 2| = |1 + 4i + 2| = |3 + 4i| = \sqrt{3^2 + 4^2} = 5$

$|\alpha^* + 2| = |1 - 4i + 2| = |3 - 4i| = \sqrt{3^2 + 4^2} = 5$

$|-7 + 2| = |-5| = 5$

so all three roots satisfy $|z + 2| = 5$.



[6]

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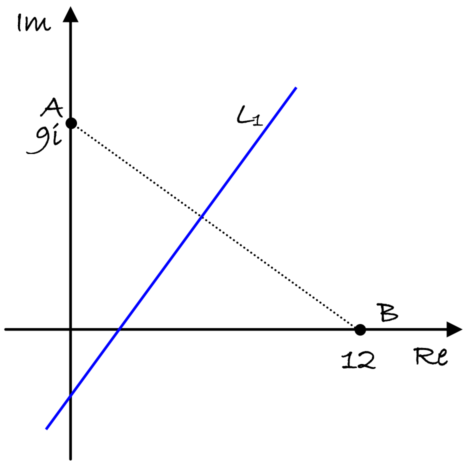
4. (i) L_1 is the perpendicular bisector of a line joining the points $z = 12$ and $z = 9i$ on the Argand diagram.



[5]

(ii) $|z - 9i| = 2|z - 12|$

$|x + iy - 9i| = 2|x + iy - 12|$

$\sqrt{x^2 + (y-9)^2} = 2\sqrt{(x-12)^2 + y^2}$

$x^2 + (y-9)^2 = 4\left((x-12)^2 + y^2\right)$

$x^2 + y^2 - 18y + 81 = 4x^2 - 96x + 576 + 4y^2$

$3x^2 - 96x + 3y^2 + 18y + 495 = 0$

$x^2 - 32x + y^2 + 6y + 165 = 0$

$(x-16)^2 - 256 + (y+3)^2 - 9 + 165 = 0$

$(x-16)^2 + (y+3)^2 = 100$

This is a circle, centre (16, -3), radius 10.

[6]

(iii) $|z - 16 + 3i| = 10$

[2]