Section 3: Invariance

Notes and Examples

These notes contain subsections on

- **Invariant points**
- **Invariant lines**

Invariant points

An **invariant point** for a transformation is a point which is mapped to itself by the transformation.

For all linear transformations, either the origin is the only invariant point, or all the invariant points lie on a straight line through the origin.

All you need to do to find the invariant points for the transformation with matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is to write the matrix equation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

write down two equations from this matrix equation

$$ax + cy = x$$
$$bx + dy = y$$

and decide whether or not these two equations are equivalent to each other. If they are, then they represent a line of invariant points, and all points on the line can be represented in terms of a parameter. If they are not equivalent, then the only invariant point is the origin.

Finding the invariant points of a transformation can give you useful information about the transformation. For example, if you know that the transformation is a reflection, then finding the line of invariant points gives you the equation of the mirror line.

Invariant lines

An **invariant line** for a transformation is a line which is mapped to itself by the transformation. It is important to understand the difference between a *line of invariant points* and an *invariant line*. On a line of invariant points, all points are mapped to themselves. However, on an invariant line, all points are mapped to a point on the line, so that the line is mapped to itself, but the individual points are not necessarily mapped to themselves.

Notice that a line of invariant points is an invariant line, but an invariant line is not necessarily a line of invariant points!

An invariant line can be found by finding the image of a general point on the line $y = mx + c$:

$$\mathbf{M}\begin{pmatrix} x \\ mx+c \end{pmatrix}$$

The image must itself satisfy the equation $y = mx + c$. This allows you to find the values of m and c, as shown in the example below.

Example 1

Find the line of invariant points and invariant lines of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

Solution

Invariant points satisfy the equation $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{aligned} 2x+y&=x \\ 2x+3y&=y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x+y&=0 \\ 2x+2y&=0 \end{aligned}$$

These two equations are equivalent

All points satisfying $x+y=0$ are invariant points.
So the line $y=-x$ is a line of invariant points.

Let the invariant lines be $y = mx + c$, so a point on the invariant line is $(x, mx + c)$.

The image of this point is $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix}$, which is $\begin{pmatrix} 2x+mx+c \\ 2x+3mx+3c \end{pmatrix}$.

This image point must lie on the line $y = mx + c$, so

$$2x+3mx+3c = m(2x+mx+c)+c$$

$$2x+3mx+3c = 2mx+m^2x+mc+c$$

$$m^2x-mx-2x+mc-2c=0$$

$$(m^2-m-2)x+c(m-2)=0$$

$$(m-2)(m+1)x+c(m-2)=0$$

This needs to be true for all values of x, so $m = 2$ or $m = -1$.

If $m = 2$, then the constant term $c(m-2)$ is also zero, so c can take any value.
Therefore all lines of the form $y = 2x + c$ are invariant lines.

If $m = -1$, then c must be zero to make the constant term $c(m-2)$ zero.
Therefore $y = -x$ is an invariant line – this is the line of invariant points found above.