

Section 3: Invariance

Notes and Examples

These notes contain subsections on

- Invariant points
- Invariant lines

Invariant points

An **invariant point** for a transformation is a point which is mapped to itself by the transformation.

For all linear transformations, either the origin is the only invariant point, or all the invariant points lie on a straight line through the origin.

All you need to do to find the invariant points for the transformation with matrix

 $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is to write the matrix equation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

write down two equations from this matrix equation

$$ax + cy = x$$
$$bx + dy = y$$

and decide whether or not these two equations are equivalent to each other. If they are, then they represent a line of invariant points, and all points on the line can be represented in terms of a parameter. If they are not equivalent, then the only invariant point is the origin.

Finding the invariant points of a transformation can give you useful information about the transformation. For example, if you know that the transformation is a reflection, then finding the line of invariant points gives you the equation of the mirror line.

Invariant lines

An **invariant line** for a transformation is a line which is mapped to itself by the transformation. It is important to understand the difference between a *line of invariant points* and an *invariant line*. On a line of invariant points, all points are mapped to themselves. However, on an invariant line, all points are mapped to a point on the line, so that the line is mapped to itself, but the individual points are not necessarily mapped to themselves.

Notice that a line of invariant points is an invariant line, but an invariant line is not necessarily a line of invariant points!



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An invariant line can be found by finding the image of a general point on the line y = mx + c:

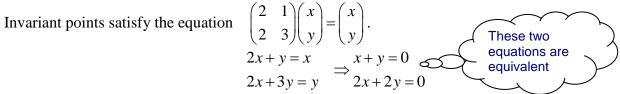
$$\mathbf{M}\begin{pmatrix}x\\mx+c\end{pmatrix}$$

The image must itself satisfy the equation y = mx + c. This allows you to find the values of *m* and *c*, as shown in the example below.

Example 1

Find the line of invariant points and invariant lines of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

Solution



All points satisfying x + y = 0 are invariant points. So the line y = -x is a line of invariant points.

Let the invariant lines be y = mx + c, so a point on the invariant line is (x, mx + c).

The image of this point is $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix}$, which is $\begin{pmatrix} 2x+mx+c \\ 2x+3mx+3c \end{pmatrix}$. This image point must lie on the line y = mx + c, so 2x + 3mx + 3c = m(2x + mx + c) + c $2x + 3mx + 3c = 2mx + m^2x + mc + c$ $m^2x - mx - 2x + mc - 2c = 0$ $(m^2 - m - 2)x + c(m - 2) = 0$ (m-2)(m+1)x + c(m-2) = 0

This needs to be true for all values of x, so m = 2 or m = -1.

If m = 2, then the constant term c(m - 2) is also zero, so c can take any value. Therefore all lines of the form y = 2x + c are invariant lines.

If m = -1, then *c* must be zero to make the constant term c(m - 2) zero. Therefore y = -x is an invariant line – this is the line of invariant points found above.

