Section 2: The inverse of a 3×3 matrix

Notes and Examples

These notes contain subsections on

In this section you will learn to find the determinant and inverse of a 3x3 matrix without using the matrix facility on a calculator. Sometimes you may need to work with matrices in which one or more element is a variable, so it is important to understand the process of finding determinants and inverses.

The minors of a 3 × 3 matrix

The minor of a particular element of a matrix is found by eliminating the row and column of that element and finding the determinant of the remaining matrix.

For a 3 × 3 matrix, the remaining matrix will be a 2 × 2 matrix.

Example 1

For the matrix $\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$

find the minors of each of the following elements:

(i) 2 (ii) 1 (iii) -2

Solution

(i) The element 2 is in row 1, column 1, so cross out this row and column:

$$\begin{pmatrix} \cancel{2} & \cancel{-1} & \cancel{4} \\ \cancel{0} & 3 & -2 \\ \cancel{-4} & 1 & -3 \end{pmatrix}$$

leaving the 2 × 2 matrix

$$\begin{pmatrix} 3 & -2 \\ 1 & -3 \end{pmatrix}$$

which has determinant $(3\times-3)-(-2\times1)=-9+2=-7$

The minor of 2 is -7.

(ii) The element 1 is in row 3, column 2, so cross out this row and column:

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

leaving the 2×2 matrix

$$\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$$

which has determinant $(2 \times -2) - (4 \times 0) = -4 - 0 = -4$

The minor of 1 is -4.

(iii) The element -2 is in row 2, column 3, so cross out this row and column:

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

leaving the 2×2 matrix

$$\begin{pmatrix} 2 & -1 \\ -4 & 1 \end{pmatrix}$$

which has determinant $(2 \times 1) - (-1 \times -4) = 2 - 4 = -2$

The minor of -2 is -2.

The cofactors of a 3 × 3 matrix

The cofactor A_{ij} of an element a_{ij} of a matrix is found by multiplying the minor of that element by $(-1)^{i+j}$. In other words, if you add together the row number and column number of an element, the result tells you whether or not to change the sign of the minor. If $i + j$ is odd, then you find the cofactor by multiplying the minor by -1 (i.e. you change its sign). If $i + j$ is even, then you find the cofactor by multiplying the minor by 1 (i.e. leave it as it is).

For a 3 × 3 matrix, the signs of $(-1)^{i+j}$ for each element are shown below. These are called the **place signs**.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

You may find it easier to remember this diagram of place signs rather than to think about the value of $(-1)^{i+j}$.

Example 2
For the matrix

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

find the cofactors of each of the following elements:
(i) 4 (ii) 0 (iii) -4

Solution

(i) The minor of 4 is $\begin{vmatrix} 0 & 3 \\ -4 & 1 \end{vmatrix} = (0\times 1)-(3\times -4) = 0+12 = 12$

The place sign of 4 is +
The cofactor of 4 is therefore 12.

(ii) The minor of 0 is $\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = (-1\times -3)-(4\times 1) = 3-4 = -1$

The place sign of 0 is –
The cofactor of 0 is therefore 1.

(iii) The minor of -4 is $\begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} = (-1\times -2)-(4\times 3) = 2-12 = -10$

The place sign of -4 is +
The cofactor of -4 is therefore -10.

Notice that the place sign does not tell you whether the cofactor is positive or negative. In (ii) of the example above, the place sign is negative but the cofactor is positive, and in (iii) the place sign is positive but the cofactor is negative.

A different method of finding cofactors

There is an alternative method of finding cofactors directly, without having to think about signs. In this method you copy the first and second columns to the right of the matrix, and copy the first and second rows (including the added columns) below the matrix. Then the cofactor of an element is simply the determinant of the 2 × 2 block directly below and to the right of that element.

Example 3 is the same as Example 2, but uses this method.

Example 3

For the matrix

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

find the cofactors of each of the following elements:

(i) 4 (ii) 0 (iii) -4

Solution

$$\begin{array}{ccccc} 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \\ -4 & 1 & -3 & -4 & 1 \\ 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \end{array}$$

Copy the first two columns here

Copy the first two rows here

(i)
$$\begin{array}{ccccc} 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \\ -4 & 1 & -3 & -4 & 1 \\ 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \end{array}$$

Cofactor of 4 $= \begin{vmatrix} 0 & 3 \\ -4 & 1 \end{vmatrix} = 0-(-12) = 12$

(ii)
$$\begin{array}{ccccc} 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \\ -4 & 1 & -3 & -4 & 1 \\ 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \end{array}$$

Cofactor of 0 $= \begin{vmatrix} 1 & -3 \\ -1 & 4 \end{vmatrix} = 4-3 = 1$

(iii)
$$\begin{array}{ccccc} 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \\ -4 & 1 & -3 & -4 & 1 \\ 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & -2 & 0 & 3 \end{array}$$

Cofactor of -4 $= \begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix} = 2-12 = -10$

Finding the determinant of a 3 × 3 matrix

The determinant of a 3 × 3 matrix can be found from the cofactors of any row or column of the matrix. Each element in that row or column is multiplied by its cofactor, and the results are added together.

For example, the determinant of a 3 × 3 matrix **A** can be found from the first column using

$$|\mathbf{A}| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

This is called **expanding by the first column**.

A similar formula applies to any row or column of the matrix.

Example 4

Find the determinant of the matrix

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix}$$

by (i) expanding by the first row
(ii) expanding by the first column.

Solution

(i) Determinant = 2 × (cofactor of 2) + -1 × (cofactor of -1) + 4 × (cofactor of 4)

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ -4 & 1 & -3 \end{pmatrix} \begin{matrix} 2 & -1 \\ 0 & 3 \\ -4 & 1 \end{matrix}$$

$$\begin{matrix} 2 & -1 & 4 & 2 & -1 \\ 0 & 3 & 2 & 0 & 3 \end{matrix}$$

The cofactor of 2 is $\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} = -9 - (-2) = -7$

The cofactor of -1 is $\begin{vmatrix} -2 & 0 \\ -3 & -4 \end{vmatrix} = 8 - 0 = 8$

The cofactor of 4 is $\begin{vmatrix} 0 & 3 \\ -4 & 1 \end{vmatrix} = 0 - (-12) = 12$

The determinant $= (2 \times -7) + (-1 \times 8) + (4 \times 12) = -14 - 8 + 48 = 26$

(ii) Determinant = 2 × (cofactor of 2) + 0 × (cofactor of 0) + -4 × (cofactor of -4)
= 2 × (cofactor of 2) + -4 × (cofactor of -4)

Notice that there will be no need to calculate the cofactor of 0, since it would just be multiplied by 0 when calculating the determinant.

The cofactor of 2 is $\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} = -9 - (-2) = -7$

The cofactor of -4 is $\begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} = 2 - 12 = -10$

$$\begin{aligned}\text{The determinant } &= (2 \times -7) + (-4 \times -10) \\ &= -14 + 40 \\ &= 26\end{aligned}$$

The determinant is the same no matter which row or column is used in the calculation. However, you can save work by choosing the row or column carefully. In part (ii) of the example above, since one of the elements in the column was zero, there was no need to calculate the cofactor for that element. So if possible you should choose a row or column which contains one or more zeros.

Example 5
Find the determinant of the matrix

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

There are two zeros in the second row, so find the determinant by expanding along the second row.

Solution
Expanding along second row: determinant = -2 × (cofactor of -2)

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 4 \end{pmatrix}\begin{matrix} 2 & -1 \\ 0 & -2 \\ 1 & 0 \end{matrix}$$

$$\begin{matrix} 2 & -1 & 3 & 2 & -1 \\ 0 & -2 & 0 & 0 & -2 \end{matrix}$$

Cofactor of -2 = $\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5$

Determinant = -2 × 5 = -10.

The method shown above for finding the determinant of a 3 × 3 matrix can be applied to square matrices of any size.

The method gives the expected result for a 2 × 2 matrix:

For the 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The minor of a is the determinant of single element d, which is d.
The place sign of a is +, so the cofactor of a is d.
The minor of b is the determinant of the single element c, which is c.
The place sign of b is –, so the cofactor of b is $-c$.

The determinant is $a \times$ (cofactor of a) + $b \times$ (cofactor of b) = $ad - bc$ as expected.

For a 4 × 4 matrix, you need to find four cofactors, each of which involves findng the determinant of a 3 × 3 matrix, so it is quite long-winded to do this by hand.

The inverse of a 3 × 3 matrix

The method for finding the inverse of a 3 × 3 matrix is as follows:

- Find the determinant of the matrix
- Find the cofactor of every element of the matrix
- Replace each element with its cofactor
- Take the transpose of this matrix (i.e. interchange the rows and columns. This matrix is called the adjugate or adjoint matrix
- Divide by the determinant to find the inverse matrix.

Again, if the determinant is zero, the inverse matrix does not exist and the matrix is singular.

The cofactors are found in the same way as they were found for the determinant, except that you need to find cofactors for all elements, not just for one row or column.

Example 6

Find the inverse of the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 4 & -2 \\ 2 & 1 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & 4 & -2 \\ 2 & 1 & 1 \end{pmatrix}\begin{matrix} 2 & 2 \\ 1 & 4 \\ 2 & 1 \end{matrix}$$

$$\begin{matrix} 2 & 2 & 0 & 2 & 2 \\ 1 & 4 & -2 & 1 & 4 \end{matrix}$$

Work out the determinant before finding any more cofactors, since if the determinant is zero there will be no point going any further!

Expanding by first row, determinant $= 2\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}$

$$= 2(4-(-2)) + 2(-4-1) = 12 - 10 = 2$$

Find the cofactor of each element by finding the determinant of the 2 × 2 block below and to the right of the element.

The matrix of cofactors is $\begin{pmatrix} 6 & -5 & -7 \\ -2 & 2 & 2 \\ -4 & 4 & 6 \end{pmatrix}$

Replace each element with its cofactor

Transpose the matrix of cofactors (i.e. interchange rows and columns)

The adjoint matrix is $\begin{pmatrix} 6 & -2 & -4 \\ -5 & 2 & 4 \\ -7 & 2 & 6 \end{pmatrix}$

The inverse matrix is $\frac{1}{2}\begin{pmatrix} 6 & -2 & -4 \\ -5 & 2 & 4 \\ -7 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -2 \\ -\frac{5}{2} & 1 & 2 \\ -\frac{7}{2} & 1 & 3 \end{pmatrix}$

Divide by the determinant to give the inverse matrix. If the determinant is large, leave it outside the matrix to avoid messy fractions.

Note that this procedure for finding an inverse matrix applies to square matrices of any size – although the alternative method for finding cofactors, by adding rows and columns, does not work for the 2 × 2 matrix.

For the general 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the minor of each element is the element diagonally opposite it. Replacing each element with its cofactor gives $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ (remember the place signs), and then taking the transpose of this matrix gives $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Dividing by the determinant gives $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

For matrices greater than 3 × 3, the same method applies, but it is of course very tedious to do it by hand. Computer software and some calculators can do this.