Section 1: Determinants and inverses

Notes and Examples

These notes contain subsections on

- The determinant of a matrix
- Inverse matrices
- Finding the inverse of a matrix
- Inverse transformations
- Matrices with zero determinant

The determinant of a matrix

The **determinant** of a matrix is important in a number of applications of matrices. In particular, it is needed to find the inverse of a matrix.

The determinant of a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $ad - bc$.

It is denoted by $\det \mathbf{A}$, $|\mathbf{A}|$, or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The determinant of a matrix is equal to the ratio of the area of image shape to the area of the object shape.

Example 1

(i) Find the determinants of each of the following matrices.

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 2 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 4 & 8 \\ -1 & -2 \end{pmatrix}$$

(ii) A shape with area 5 square units is transformed by matrix A. What is the area of its image?

Solution

(i) $|\mathbf{A}| = (6\times3)-(4\times2)=18-8=10$

$|\mathbf{B}| = (1\times3)-(-2\times2)=3+4=7$

$|\mathbf{C}| = =(4\times-2)-(8\times-1)=-8+8=0$

(ii) Area of image $=5\times10=50$ square units.

A matrix whose determinant is zero, like matrix **C** in Example 1, is called a **singular matrix**. Matrices **A** and **B** in Example 1 are **non-singular**.
In some applications of matrices it is very important to know whether a matrix is singular or not. In particular, a singular matrix has no inverse.

Example 2

The matrix $\mathbf{M} = \begin{pmatrix} 1-k & 2 \\ -1 & 4-k \end{pmatrix}$ is singular. Find the possible values of k.

Solution

$$\begin{aligned} |\mathbf{M}| &= (1-k)(4-k)-(2\times -1) \\ &= 4-5k+k^2+2 \\ &= k^2-5k+6 \end{aligned}$$

Since **M** is singular, |**M**| = 0

$$k^2-5k+6=0$$

$$(k-2)(k-3)=0$$

$$k=2 \text{ or } k=3$$

The determinant is defined for any square matrix. At this stage, you can find the determinant of a 3x3 matrix using the matrix facility on your calculator. Later in A level Further Mathematics, you will see how to do this without using a calculator.

If you apply one transformation represented by **N** followed by another one represented by **M**, at the first stage the area of the object is multiplied by det **N**, and at the second stage the area is multiplied by det **M**. So the area factor of the combined transformation is det **M** x det **N**.
Hence det(**MN**) = det **M** x det **N**.

Inverse matrices

The inverse of a transformation undoes the effect of the original transformation.

- The inverse of a rotation is a rotation through the same angle about the same point but in the opposite direction.
- The inverse of a reflection is another reflection in the same line.
- The inverse of an enlargement with scale factor k is an enlargement with scale factor $\frac{1}{k}$.

For a $n \times n$ matrix **A**, the inverse matrix (also a $n \times n$ matrix) is denoted $\mathbf{A}^{-1}$, and satisfies the relationship

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

This makes sense – the identity matrix maps a point to itself, and a transformation followed by its inverse maps a point back to its original position. So a matrix multiplied by its inverse gives the identity matrix.

You can see that the inverse of a matrix is rather like the reciprocal of a number when dealing with real numbers. Just as multiplying a number by its reciprocal always gives 1, multiplying a matrix by its inverse always gives the identity matrix.

Example 3

(i) Show that $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$.

(ii) Show that $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix}$ is the inverse of $\begin{pmatrix} -2 & -2 & 1 \\ -2 & -1 & 1 \\ -3 & -3 & 1 \end{pmatrix}$

Solution

(i) $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$.

(ii) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix}\begin{pmatrix} -2 & -2 & 1 \\ -2 & -1 & 1 \\ -3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -2 & -2 & 1 \\ -2 & -1 & 1 \\ -3 & -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

so $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix}$ is the inverse of $\begin{pmatrix} -2 & -2 & 1 \\ -2 & -1 & 1 \\ -3 & -3 & 1 \end{pmatrix}$

Finding the inverse of a matrix

To find the inverse of a 2×2 matrix:

- Interchange the elements on the leading diagonal
- Change the signs of the other elements
- Multiply by the reciprocal of the determinant

So the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If the determinant is zero, the inverse matrix does not exist.

Example 4

Find, if possible, the inverses of the matrices

(i) $\begin{pmatrix} 7 & 8 \\ 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$

Solution

Interchange the elements on the leading diagonal and change the sign of the other two elements.

(i) $\begin{vmatrix} 7 & 8 \\ 3 & 4 \end{vmatrix} = (7 \times 4) - (8 \times 3) = 28 - 24 = 4$

The inverse of $\begin{pmatrix} 7 & 8 \\ 3 & 4 \end{pmatrix}$ is $\frac{1}{4}\begin{pmatrix} 4 & -8 \\ -3 & 7 \end{pmatrix}$ which may be written as $\begin{pmatrix} 1 & -2 \\ -\frac{3}{4} & \frac{7}{4} \end{pmatrix}$

(ii) $\begin{vmatrix} 4 & 5 \\ -2 & -3 \end{vmatrix} = (4 \times -3) - (5 \times -2) = -12 + 10 = -2$

The inverse of $\begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$ is $-\frac{1}{2}\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$ which may be written as $\begin{pmatrix} \frac{3}{2} & \frac{5}{2} \\ -1 & -2 \end{pmatrix}$

Interchange the elements on the leading diagonal and change the sign of the other two elements.

(iii) $\begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix}$ is $(2 \times 4) - (8 \times 1) = 8 - 8 = 0$

So $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$ has no inverse.

You can find the inverse of a 3x3 matrix using the matrix facility on your calculator. Later in A level Further Mathematics you will learn to do this without a calculator.

Make sure that you remember the rule for the inverse of a product:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

It is easy to understand if you think of it in terms of transformations: if you want to undo two transformations, you must undo the second one before undoing the first one.

Matrices with zero determinant

A matrix with zero determinant maps all point onto a straight line.

If a matrix has zero determinant, it has no inverse. Algebraically, you can see that this is the case because you cannot divide by zero. However, it also makes sense in the context of transformations because a singular matrix transforms all points onto a single line (or onto a single point, the origin, in the case of the zero matrix). This means that each image point could be the image of an infinite number of different object points, so it is impossible to distinguish which object point each image point came from. This means it is impossible to "undo" the transformation.

Note also that the area of a line or point is zero, and so det **T** = 0 for a singular matrix can still be thought of as the area factor.