Section 3: Solving simultaneous differential equations

Section test

1. A polluted river flows into a lake, A, at the rate of 200 km³ per year. 20000 kg of chemical pollution enters the lake each year. Water flows out of lake A at the same rate and into a second lake, B. A second river of clean water also flows into lake B at a rate of 300 km³ per year. Water flows out of lake B at a rate of 500 km³ per year, so that the volume of water in each lake remains constant. Lake A has a volume of 4000 km³ and lake B has a volume of 2500 km³.
   Initially both lakes contain no pollution.
   Let x be the level of pollution in lake A, in kg/km³.
   Let y be the level of pollution in lake B, in kg/km³.
   Find expressions for $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ to model this situation.

2. A system of differential equations is given by
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - 3y - 1$$
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y - 3$$
   The system can be reduced to the second order differential equation
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = c \ .$$
   Find the values of a, b and c.

3. A system of differential equations is given by
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y + 1$$
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x + y + 2$$
   The system can be reduced to the second order differential equation
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = c \ .$$
   Find the values of a, b and c.

   What happens to the value of x as t becomes very large?
   (a) It approaches zero without oscillating
   (b) It approaches a fixed non-zero value without oscillating
   (c) It oscillates towards zero
   (c) It oscillates towards a fixed non-zero value
   (d) It oscillates, with the oscillations becoming larger and larger
   (e) It becomes numerically very large, without oscillating

Edexcel FM Second order DEs 3 section test solutions

4. A system of differential equations is given by

$$\frac{dx}{dt} = -x - y + 1$$

$$\frac{dy}{dt} = 2x - 3y + 3$$

The system can be reduced to the second order differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = c \quad .$$

Find the values of a, b and c.

What happens to the value of x as t becomes very large?

(a) It approaches zero without oscillating
(b) It approaches a fixed non-zero value without oscillating
(c) It oscillates towards zero
(c) It oscillates towards a fixed non-zero value
(d) It oscillates, with the oscillations becoming larger and larger
(e) It becomes numerically very large, without oscillating

5. The system of differential equations

$$\frac{dx}{dt} = -3x + 2y$$

$$\frac{dy}{dt} = -2x + y$$

with initial conditions $x = 1$ and $y = 2$ when $t = 0$
has the particular solution

$$x = (a + bt)e^{kt}, \quad y = (c + d)e^{kt}$$

Solve the system of equations to find the values of a, b, c, d and k.

6. The system of differential equations

$$\frac{dx}{dt} = x - y + 3$$

$$\frac{dy}{dt} = 5x - y - 1$$

with initial conditions $x = 0$ and $y = 1$ when $t = 0$
has the particular solution

$$x = a + b\sin 2t + c\cos 2t$$

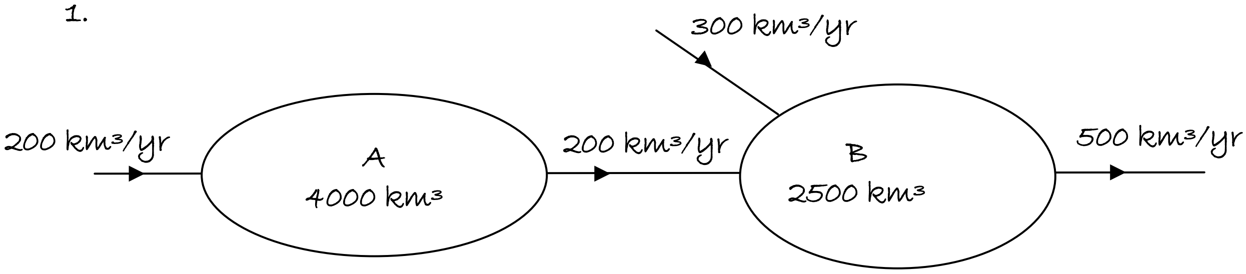
$$y = p + q\sin 2t + r\cos 2t$$

Solve the system of equations to find the values of a, b, c, p, q and r.

Edexcel FM Second order DEs 3 section test solutions

Section test solutions

1.



Rate of input of pollution for A = 20000.
Rate of output of pollution for A = $200x$

Rate of change of pollution for A is $\frac{dx}{dt} = \frac{20000 - 200x}{4000}$

$$\frac{dx}{dt} = 5 - 0.05x$$

Rate of input of pollution for B = $200x$.
Rate of output of pollution for B = $500y$

Rate of change of pollution for B is $\frac{dy}{dt} = \frac{200x - 500y}{2500}$

$$\frac{dy}{dt} = 0.08x - 0.2y$$

2. $\frac{dx}{dt} = 2x - 3y - 1 \quad \Rightarrow y = \frac{1}{3}\left(2x - 1 - \frac{dx}{dt}\right)$

$$\frac{dy}{dt} = \frac{1}{3}\left(2\frac{dx}{dt} - \frac{d^2x}{dt^2}\right)$$

Substituting for y and $\frac{dy}{dt}$ in $\frac{dy}{dt} = x + y - 3$:

$$\frac{1}{3}\left(2\frac{dx}{dt} - \frac{d^2x}{dt^2}\right) = x + \frac{1}{3}\left(2x - 1 - \frac{dx}{dt}\right) - 3$$

y'' $\quad 2\frac{dx}{dt} - \frac{d^2x}{dt^2} = 3x + 2x - 1 - \frac{dx}{dt} - 9$

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 5x = 10$$

Edexcel FM Second order DEs 3 section test solutions

3. $\frac{dx}{dt} = -4x + 2y + 1 \quad \Rightarrow \quad y = \frac{1}{2}\left(\frac{dx}{dt} + 4x - 1\right)$

$$\frac{dy}{dt} = \frac{1}{2}\left(\frac{d^2x}{dt^2} + 4\frac{dx}{dt}\right)$$

Substituting for y and $\frac{dy}{dt}$ into $\frac{dy}{dt} = -3x + y + 2$:

$$\frac{1}{2}\left(\frac{d^2x}{dt^2} + 4\frac{dx}{dt}\right) = -3x + \frac{1}{2}\left(\frac{dx}{dt} + 4x - 1\right) + 2$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = -6x + \frac{dx}{dt} + 4x - 1 + 4$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 3$$

Auxiliary equation is $\lambda^2 + 3\lambda + 2 = 0$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1 \text{ or } -2$$

Complementary function is $x = Ae^{-t} + Be^{-2t}$

Particular integral is $x = c$

Substituting into differential equation: $2c = 3 \quad \Rightarrow c = \frac{3}{2}$

General solution is $x = Ae^{-t} + Be^{-2t} + \frac{3}{2}$.

As $t \to \infty$, $x \to \frac{3}{2}$, so x approaches a fixed value as $t \to \infty$, without oscillating

4. $\frac{dx}{dt} = -x - y + 1 \quad \Rightarrow \quad y = -\frac{dx}{dt} - x + 1$

$$\frac{dy}{dt} = -\frac{d^2x}{dt^2} - \frac{dx}{dt}$$

Substituting for y and $\frac{dy}{dt}$ into $\frac{dy}{dt} = 2x - 3y + 3$:

$$-\frac{d^2x}{dt^2} - \frac{dx}{dt} = 2x - 3\left(-\frac{dx}{dt} - x + 1\right) + 3$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0$$

Auxiliary equation is $\lambda^2 + 4\lambda + 5 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

General solution is $x = e^{-2t}(A\sin t + B\cos t)$

As $t \to \infty$, x oscillates towards zero.

Edexcel FM Second order DEs 3 section test solutions

5. $\frac{dx}{dt} = -3x + 2y \quad \Rightarrow y = \frac{1}{2}\left(\frac{dx}{dt} + 3x\right)$

$$\frac{dy}{dt} = \frac{1}{2}\left(\frac{d^2x}{dt^2} + 3\frac{dx}{dt}\right)$$

Substituting for y and $\frac{dy}{dt}$ in $\frac{dy}{dt} = -2x + y$:

$$\frac{1}{2}\left(\frac{d^2x}{dt^2} + 3\frac{dx}{dt}\right) = -2x + \frac{1}{2}\left(\frac{dx}{dt} + 3x\right)$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = -4x + \frac{dx}{dt} + 3x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

Auxiliary equation is $\lambda^2 + 2\lambda + 1 = 0$

$$(\lambda + 1)^2 = 0$$

$$\lambda = -1$$

Complementary function is $x = (A + Bt)e^{-t}$

When $t = 0, x = 1 \Rightarrow 1 = A$

$$\frac{dx}{dt} = -(A + Bt)e^{-t} + Be^{-t}$$

When $t = 0, x = 1$ and $y = 2 \Rightarrow \frac{dx}{dt} = -3 + 4 = 1$

$1 = -A + B \quad \Rightarrow B = 2$

Particular solution for x is $x = (1 + 2t)e^{-t}$

$$\frac{dx}{dt} = -(1 + 2t)e^{-t} + 2e^{-t} = (1 - 2t)e^{-t}$$

$$y = \frac{1}{2}\left(\frac{dx}{dt} + 3x\right)$$
$$= \tfrac{1}{2}\left((1 - 2t)e^{-t} + 3(1 + 2t)e^{-t}\right)$$
$$= \tfrac{1}{2}(4 + 4t)e^{-t}$$
$$= (2 + 2t)e^{-t}$$

6. $\frac{dx}{dt} = x - y + 3 \quad \Rightarrow y = x - \frac{dx}{dt} + 3$

$$\frac{dy}{dt} = \frac{dx}{dt} - \frac{d^2x}{dt^2}$$

Substituting for y and $\frac{dy}{dt}$ in $\frac{dy}{dt} = 5x - y - 1$:

Edexcel FM Second order DEs 3 section test solutions

$$\frac{dx}{dt} - \frac{d^2x}{dt^2} = 5x - \left(x - \frac{dx}{dt} + 3\right) - 1$$

$$\frac{d^2x}{dt^2} + 4x = 4$$

Auxiliary equation is $\lambda^2 + 4 = 0$

$$\lambda = \pm 2i$$

Complementary function is $x = A\sin 2t + B\cos 2t$

Particular integral is $x = c$

Substituting into differential equation $\Rightarrow 4c = 4 \quad \Rightarrow c = 1$

General solution is $x = A\sin 2t + B\cos 2t + 1$

When $t = 0, x = 0 \Rightarrow 0 = B + 1 \quad \Rightarrow B = -1$

$$\frac{dx}{dt} = 2A\cos 2t - 2B\sin 2t$$

When $t = 0, x = 0$ and $y = 1 \Rightarrow \frac{dx}{dt} = 0 - 1 + 3 = 2$

$2 = 2A \quad \Rightarrow A = 1$

Particular solution for x is $x = \sin 2t - \cos 2t + 1$

$$\frac{dx}{dt} = 2\cos 2t + 2\sin 2t$$

$$y = x - \frac{dx}{dt} + 3$$

$$= \sin 2t - \cos 2t + 1 - (2\cos 2t + 2\sin 2t) + 3$$

$$= -\sin 2t - 3\cos 2t + 4$$