Section 2: Non-homogeneous differential equations

Section test

1. Find a particular integral for the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 - 1$$

2. Find a particular integral for the differential equation

$$2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = e^{-2t}$$

3. A particular integral for the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{2x}$$

is given by

(a) $\frac{1}{4}xe^{2x}$ (b) $\frac{1}{7}e^{2x}$

(c) $\frac{1}{5}xe^{2x}$ (d) $-\frac{1}{4}e^{2x}$

4. Find the particular integral for the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = \sin x.$$

5. The general solution of the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = -e^{-t}$$

is given by:

(a) $x = Ae^{4t} + Be^{-t} + \frac{1}{6}e^{-t}$ (b) $x = Ae^{-4t} + Be^{t} + \frac{1}{6}e^{-t}$

(c) $x = Ae^{-4t} + Be^{t} - \frac{1}{3}e^{-t}$ (d) $x = Ae^{4t} + Be^{-t} - \frac{1}{3}e^{-t}$

6. The general solution of the differential equation

$$\frac{d^2x}{dt^2} + x = \cos 2t$$

is given by:

(a) $x = A\sin t + B\cos t - \frac{1}{3}\sin 2t$ (b) $x = A\sin t + B\cos t - \frac{1}{3}\cos 2t$

(c) $x = A\sin t + B\cos t - \frac{1}{5}\cos 2t$ (d) $x = A\sin t + B\cos t - \frac{1}{5}\sin 2t$

7. The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 5x - 3$$

is given by:

(a) $y = e^{-x}(A\sin 2x + B\cos 2x) + 5x - 8$ (b) $y = e^{x}(A\sin 2x + B\cos 2x) + x - 1$

(c) $y = e^{-x}(A\sin 2x + B\cos 2x) + x - 1$ (d) $y = e^{x}(A\sin 2x + B\cos 2x) + 5x - 8$

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8. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\cos x$$

for which $y = 0$ and $\frac{dy}{dx} = 4$ when $x = 0$.

9. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 8e^{3x}$$

for which $y = 3$ and $\frac{dy}{dx} = 3$ when $x = 0$.

10. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2$$

for which $y = 1$ and $\frac{dy}{dx} = -4$ when $x = 0$.

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Solutions to section test

1. Particular integral is $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

Substituting into $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 - 1$

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 2x^2 - 1$$

$$2ax^2 + (-6a + 2b)x + 2a - 3b + 2c = 2x^2 - 1$$

Equating coefficients of x^2: $2a = 2 \Rightarrow a = 1$

Equating coefficients of x: $-6a + 2b = 0 \Rightarrow b = 3a = 3$

Equating constant terms: $2a - 3b + 2c = -1 \Rightarrow 2 - 9 + 2c = -1$

$\Rightarrow c = 3$

The particular integral is $y = x^2 + 3x + 3$.

2. The particular integral is $x = ae^{-2t}$

$$\frac{dx}{dt} = -2ae^{-2t}$$

$$\frac{d^2x}{dt^2} = 4ae^{-2t}$$

Substituting into $2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = e^{-2t}$

$$8ae^{-2t} - 2ae^{-2t} + 2ae^{-2t} = e^{-2t}$$

$$8ae^{-2t} = e^{-2t}$$

$$a = \tfrac{1}{8}$$

The particular integral is $x = \tfrac{1}{8}e^{-2t}$

3. The auxiliary equation is $\lambda^2 + \lambda - 6 = 0$

$\lambda = 2$ is a solution of the auxiliary equation.

So Ae^{2x} is one term in the complementary function. This affects the form of the particular integral.

The particular integral is therefore of the form $y = axe^{2x}$

$$\frac{dy}{dx} = 2axe^{2x} + ae^{2x}$$

$$\frac{d^2y}{dx^2} = 4axe^{2x} + 2ae^{2x} + 2ae^{2x} = 4axe^{2x} + 4ae^{2x}$$

Substituting into the differential equation:

$$4axe^{2x} + 4ae^{2x} + 2axe^{2x} + ae^{2x} - 6axe^{2x} = e^{2x}$$
$$5ae^{2x} = e^{2x}$$
$$a = \tfrac{1}{5}$$

The particular integral is $y = \tfrac{1}{5}xe^{2x}$

4. The particular integral is $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$
$$\frac{d^2y}{dx^2} = -a\sin x - b\cos x$$

Substituting into $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2y = \sin x$

$$-a\sin x - b\cos x - (a\cos x - b\sin x) + 2(a\sin x + b\cos x) = \sin x$$
$$(a+b)\sin x + (b-a)\cos x = \sin x$$

Equating coefficients of cos x: $b - a = 0 \quad \Rightarrow a = b$

Equating coefficients of sin x: $a + b = 1 \quad \Rightarrow 2a = 1 \quad \Rightarrow a = \tfrac{1}{2}, b = \tfrac{1}{2}$

The particular integral is $y = \tfrac{1}{2}\sin x + \tfrac{1}{2}\cos x$

5. $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} - 4x = -e^{-t}$

Auxiliary equation: $m^2 + 3m - 4 = 0$

$$(m+4)(m-1) = 0$$
$$m = -4 \text{ or } m = 1$$

Complementary function is $x = Ae^{-4t} + Be^{t}$

Particular integral is $x = ae^{-t}$

$$\frac{dx}{dt} = -ae^{-t}$$
$$\frac{d^2x}{dt^2} = ae^{-t}$$

Substituting into $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} - 4x = -e^{-t}$

$$ae^{-t} - 3ae^{-t} - 4ae^{-t} = -e^{t}$$
$$-6ae^{-t} = -e^{-t}$$
$$a = \tfrac{1}{6}$$

The general solution is $x = Ae^{-4t} + Be^{t} + \tfrac{1}{6}e^{-t}$.

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6. $\dfrac{d^2x}{dt^2}+x=\cos 2t$

Auxiliary equation is $m^2+1=0$

$$m=\pm i$$

Complementary function is $x=A\cos t+B\sin t$

Particular integral is $x=a\sin 2t+b\cos 2t$

$$\frac{dx}{dt}=2a\cos 2t-2b\sin 2t$$

$$\frac{d^2x}{dt^2}=-4a\sin 2t-4b\cos 2t$$

Substituting into $\dfrac{d^2x}{dt^2}+x=\cos 2t$

$$-4a\sin 2t-4b\cos 2t+a\sin 2t+b\cos 2t=\cos 2t$$

$$-3a\sin 2t-3b\cos 2t=\cos 2t$$

Equating coefficients of cos 2t $\Rightarrow -3b=1 \quad \Rightarrow b=-\tfrac{1}{3}$

Equating coefficients of sin 2t $\Rightarrow -3a=0 \quad \Rightarrow a=0$

General solution is $x=A\cos t+B\sin t-\tfrac{1}{3}\cos 2t$

7. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+5y=5x-3$

Auxiliary equation is $m^2+2m+5=0$

$$m=\frac{-2\pm\sqrt{4-4\times 1\times 5}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4i}{2}$$

$$m=-1\pm 2i$$

Complementary function is $y=e^{-x}(A\cos 2x+B\sin 2x)$

Particular integral is $y=ax+b$

$$\frac{dy}{dx}=a$$

$$\frac{d^2y}{dx^2}=0$$

Substituting into $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+5y=5x-3$

$$2a+5(ax+b)=5x-3$$

$$5ax+2a+5b=5x-3$$

Equating coefficients of x: $\quad 5a=5 \quad \Rightarrow a=1$

Equating constant terms: $\quad 2a+5b=-3 \quad \Rightarrow 2+5b=-3 \quad \Rightarrow b=-1$

General solution is $y=e^{-x}(A\cos 2x+B\sin 2x)+x-1$

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8. Auxiliary equation is $\lambda^2 + 2\lambda + 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$\lambda = -1 \pm i$$

Complementary function is $y = e^{-x}(A\sin x + B\cos x)$

The particular integral is of the form $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$

$$\frac{d^2y}{dx^2} = -a\sin x - b\cos x$$

Substituting into differential equation:

$$-a\sin x - b\cos x + 2(a\cos x - b\sin x) + 2(a\sin x + b\cos x) = 5\cos x$$

$$(a - 2b)\sin x + (2a + b)\cos x = 5\cos x$$

Equating coefficients of sin x $\Rightarrow a - 2b = 0 \Rightarrow a = 2b$

Equating coefficients of cos x $\Rightarrow 2a + b = 5 \Rightarrow 4b + b = 5 \Rightarrow b = 1, a = 2$

General solution is $y = e^{-x}(A\sin x + B\cos x) + 2\sin x + \cos x$

When $x = 0$, $y = 0 \Rightarrow 0 = B + 1 \Rightarrow B = -1$

$$\frac{dy}{dx} = -e^{-x}(A\sin x + B\cos x) + e^{-x}(A\cos x - B\sin x) + 2\cos x - \sin x$$

When $x = 0$, $\frac{dy}{dx} = 4 \Rightarrow 4 = -B + A + 2 \Rightarrow A = 2 + B \Rightarrow A = 1$

Particular solution is $y = e^{-x}(\sin x - \cos x) + 2\sin x + \cos x$

9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 8e^{3x}$

Auxiliary equation: $m^2 - 2m - 3 = 0$

$$(m - 3)(m + 1) = 0$$

$$m = 3 \text{ or } m = -1$$

Complementary function is $y = Ae^{3x} + Be^{-x}$

Particular integral is $y = axe^{3x}$

> The right-hand side is of the same form as part of the complementary function, so the particular integral must be of this form.

$$\frac{dy}{dx} = ae^{3x} + 3axe^{3x}$$

$$\frac{d^2y}{dx^2} = 3ae^{3x} + 3ae^{3x} + 9axe^{3x} = 6ae^{x} + 9axe^{3x}$$

Substituting into $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 8e^{3x}$

$$6ae^{3x} + 9axe^{3x} - 2(ae^{3x} + 3axe^{3x}) - 3axe^{3x} = 8e^{3x}$$

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$(6-2)ae^{3x}+(9-6-3)xae^{x}=8e^{3x}$

$4ae^{3x}=8e^{3x}$

$a=2$

General solution is $y=Ae^{3x}+Be^{-x}+2xe^{3x}$

When $x=0$, $y=3$ $\Rightarrow A+B=3$

$\frac{dy}{dx}=3Ae^{3x}-Be^{-x}+2e^{3x}+6xe^{3x}$

When $x=0$, $\frac{dy}{dx}=3$ $\Rightarrow 3A-B+2=3$ $\Rightarrow 3A-B=1$

$A+B=3$

$3A-B=1$

Adding: $4A=4$ $\Rightarrow A=1, B=2$

Particular solution is $y=e^{3x}+2e^{-x}+2xe^{3x}$.

10. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2$

Auxiliary equation is $m^2-2m+1=0$

$(m-1)^2=0$

$m=1$

Complementary function is $y=(A+Bx)e^{x}$

Particular integral is $y=ax^2+bx+c$

$\frac{dy}{dx}=2ax+b$

$\frac{d^2y}{dx^2}=2a$

Substituting into $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2$

$2a-2(2ax+b)+ax^2+bx+c=x^2$

$ax^2+(-4a+b)x+2a-2b+c=x^2$

Equating coefficients of x^2: $a=1$

Equating coefficients of x: $-4a+b=0$ $\Rightarrow b=4$

Equating constant terms: $2a-2b+c=0$ $\Rightarrow c=6$

General solution is $y=(A+Bx)e^{x}+x^2+4x+6$

When $x=0$, $y=1$ $\Rightarrow 1=A+6$ $\Rightarrow A=-5$

$\frac{dy}{dx}=Be^{x}+(A+Bx)e^{x}+2x+4$

When $x=0$, $\frac{dy}{dx}=-4$ $\Rightarrow -4=B+A+4$ $\Rightarrow B=-3$

Particular solution is $y=(-5-3x)e^{x}+x^2+4x+6$.