Section 3: Solving simultaneous differential equations

Notes and Examples

These notes contain subsections on
- **Simultaneous linear differential equations**
- **Eliminating a variable**

Simultaneous linear differential equations

Many real-life situations that can be modelled by differential equations involve a large number of inter-related variables. In this section you will look at models that involve two simultaneous first order linear differential equations.

For example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 4y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

These can be solved by eliminating one of the variables to form a single second order differential equation.

An example of this type of situation is a predator-prey model. The rate of change of the population of predators depends on the size of its population, but also depends on the size of the prey population. Similarly, the rate of change of the prey population depends on its own size, but also on the size of the population of predators. You can probably see that this situation could develop in different ways depending on the initial conditions: if the predator population becomes very large, the prey population is likely to diminish, but that would result in the predator population diminishing due to insufficient food, allowing the prey population to increase again. In some circumstances, either population could die out, but alternatively they may reach an equilibrium.

In most real situations, of course, there are more factors involved (e.g. alternative foods available to the predators) so the model might not be very realistic.

Eliminating a variable

It is possible to end up going round in circles when trying to eliminate a variable from a pair of simultaneous differential equations! For this reason it is usually best to learn a standard pattern of elimination and use it consistently.

Two possible approaches to eliminating y from the equations

$$\frac{dx}{dt} = ax + by + c \qquad (1)$$
$$\frac{dy}{dt} = dx + ey + f \qquad (2)$$

are given below. Remember that you are differentiating with respect to t, not x!

Approach 1

- Rearrange (1) to give y in terms of x and $\frac{dx}{dt}$
- Differentiate with respect to t to give $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$
- Substitute your expressions for y and $\frac{dy}{dt}$ into equation (2), and rearrange to give a second order differential equation in x.

Approach 2

- Differentiate equation (1) with respect to t to give $\frac{d^2x}{dt^2}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
- Use equation (2) to substitute for $\frac{dy}{dt}$. This gives an equation for $\frac{d^2x}{dt^2}$ in terms of x, y and $\frac{dx}{dt}$.
- Rearrange the original equation (1) to give y in terms of x and $\frac{dx}{dt}$, and substitute this into the new equation. Rearrange to give a second order differential equation in x.

Of course, similar approaches can be used if you wish to eliminate x instead of y.

The example below uses Approach 1.

Example 1
Find the particular solution of the equations

$$\frac{dx}{dt} = 4x - 2y$$
$$\frac{dy}{dt} = 3x - y$$

for which $x = 2$ and $y = 4$ when $t = 0$.

Solution

Rearranging the first equation: $y = 2x - \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$

Differentiate: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

Substituting into second equation: $2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 3x - \left(2x - \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}\right)$

$$4\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 6x - 4x + \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$$

Auxiliary equation: $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1 \text{ or } 2$$

General solution is $x = A\mathrm{e}^t + B\mathrm{e}^{2t}$

So $\frac{\mathrm{d}x}{\mathrm{d}t} = A\mathrm{e}^t + 2B\mathrm{e}^{2t}$

Substituting into the first equation: $y = 2x - \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$

$$y = 2\left(A\mathrm{e}^t + B\mathrm{e}^{2t}\right) - \tfrac{1}{2}(A\mathrm{e}^t + 2B\mathrm{e}^{2t})$$

$$y = \tfrac{3}{2}A\mathrm{e}^t + B\mathrm{e}^{2t}$$

So the general solution is

$$x = A\mathrm{e}^t + B\mathrm{e}^{2t}$$
$$y = \tfrac{3}{2}A\mathrm{e}^t + B\mathrm{e}^{2t}$$

When $t = 0, x = 2 \Rightarrow 2 = A + B$

When $t = 0, y = 4 \Rightarrow 4 = \tfrac{3}{2}A + B$

Solving these equations gives $A = 4, B = -2$

So the particular solution is

$$x = 4\mathrm{e}^t - 2\mathrm{e}^{2t}$$
$$y = 6\mathrm{e}^t - 2\mathrm{e}^{2t}$$

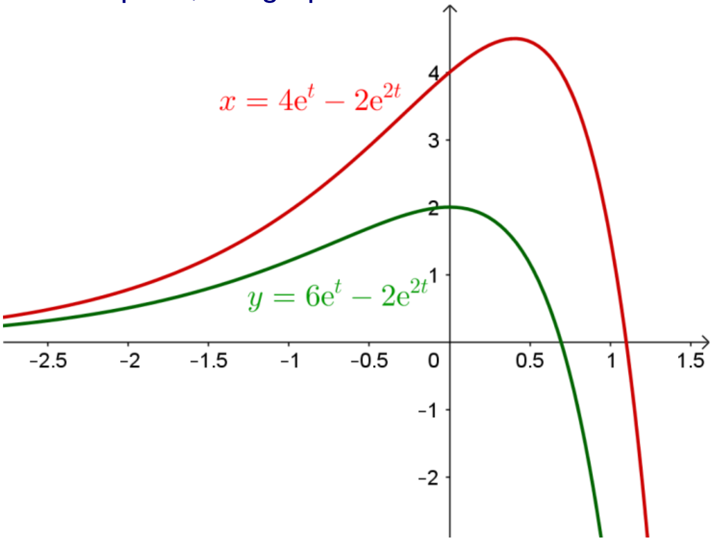
Notice that the complementary functions for the two variables have the same form as each other. This is always the case. In other words, whichever variable you choose to eliminate, you will get the same auxiliary equation to solve.

You can verify this by using the general system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + c$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = dx + ey + f$$

Eliminate y to obtain a second order differential equation in x and write down the auxiliary equation.
Then eliminate x from the original differential equations to obtain a second order differential equation in y and write down the auxiliary equation. The two auxiliary equations should be the same.

In Example 1, the graph of the solution curves is shown below.



If this situation represented a real-life scenario involving populations, the values of x, y and t must all be greater than or equal to 0. This means the graphs would start at $t = 0$ and would stop when they reached the time axis, signifying one of the populations dying out. This, of course, would change the model. So the population represented by y would die out, and at that point a new model would be needed for the population represented by x.