Section 3: Solving systems of differential equations

Exercise level 2

1. The number of deer in a forest and the amount of suitable food growing there are interdependent. A possible model for this is given by the differential equation system

$$\frac{dx}{dt} = -5x + 6y \quad \frac{dy}{dt} = -3x + y$$

where x is the number of deer in thousands, and y is the amount of food in suitable units and t is the time in years.

(i) Find a second order differential equation for x.
(ii) Find an expression for the number of deer after t years.
(iii) Determine whether the model predicts that deer population is sustainable in the long term or not.

2. Equal numbers of each of competing species are introduced to an island. The numbers of Species A and Species B are x and y respectively and can be modelled by the system of differential equations, where t is the time is years since their introduction.

$$100\frac{dx}{dt} = 2x - 12y \quad 100\frac{dy}{dt} = -x + y$$

(i) By eliminating x, find a second order differential equation for y which the model satisfies.
(ii) Find the general solution of the differential equation from part (i).
(iii) Initially there are 700 animals of each species. Find expressions for x and y at time t.
(iv) Determine whether either species will die out.
(v) Investigate different starting values to see whether extinction is inevitable.

3. As particle moves, its position in the x-y plane at time t s is given by $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ m.

The displacement satisfies the system of differential equations

$$\frac{dx}{dt} = 5x - 6y \quad \frac{dy}{dt} = 3x - y$$

(i) Find the general solution of the system of differential equations.
(ii) The initial position of the particle is $\begin{pmatrix} 1.5 \\ 0 \end{pmatrix}$.

Find an expression for the displacement vector of the particle at time t.

4. A particle moves in the x-y plane such that its coordinates (x, y) in metres at time t seconds satisfy the differential equations

$$\frac{dx}{dt} = x - 4y + a \quad \frac{dy}{dt} = 2x - 5y + b$$

(i) Find expressions for x and y at time t.
(ii) You are given that $x = 0$ and $y = 0$ when $t = 0$.
Describe the long-term behaviour of the system.