Section 2: Non-homogeneous differential equations

Notes and Examples

These notes contain subsections on

- The complementary function and particular integral
- Finding particular integrals
- Finding general and particular solutions

The complementary function and particular integral

So far, all the second order differential equations you have met have been of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0.$$

These are called homogeneous differential equations.

You will now look at differential equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}(x).$$

These are called non-homogeneous differential equations. You will be looking at equations where $\mathrm{f}(x)$ is a simple polynomial, exponential or trigonometric function.

To solve a non-homogeneous differential equation, you start by finding the general solution to the associated homogeneous differential equation, in the same way that you did in section 1. This is called the **complementary function**.

You then need to find a function which satisfies the full differential equation. This is called the **particular integral** and it is added to the complementary function to form the general solution to the non-homogeneous equation.

The form of the particular solution is related to the form of the function $\mathrm{f}(x)$. By expressing it in terms of one or more unknown constants, and substituting it into the differential equation, you can find the unknown constants.

Finding particular integrals

The table below illustrates all the forms of particular integrals that you could meet.

Function f(x)	**Particular integral**
Constant term	c
Linear function	$ax + b$
Quadratic function	$ax^2 + bx + c$
Exponential function involving e^{px}	$k\mathrm{e}^{px}$
Function involving $\sin px$ and / or $\cos px$	$a\cos px + b\sin px$

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Here is an example, illustrating an important point about the form of the particular integral.

Example 1
Find a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x^2$$

Solution

Since the function on the right-hand side is quadratic, use the trial function

$$y = ax^2 + bx + c$$
$$\frac{dy}{dx} = 2ax + b$$
$$\frac{d^2y}{dx^2} = 2a$$

Notice that even though the function on the right-hand side involves only a term in x^2, the trial function must also involve a term in x and a constant term, as these will appear when differentiating.

Substituting these into the differential equation:

$$2a - (2ax + b) - 2(ax^2 + bx + c) = x^2$$
$$-2ax^2 + (-2a - 2b)x + (2a - b - 2c) = x^2$$

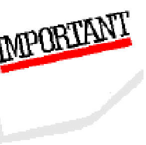
Equating coefficients of x^2: $-2a = 1 \Rightarrow a = -\frac{1}{2}$

Equating coefficients of x: $-2a - 2b = 0 \Rightarrow b = -a = \frac{1}{2}$

Equating constant terms: $2a - b - 2c = 0 \Rightarrow c = \frac{1}{2}(2a - b) = -\frac{3}{4}$

The particular integral is $y = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

A similar situation may arise when the particular integral involves trig functions. If the right-hand side of the differential equation is $\cos 2x$, but the particular integral must also involve $\sin 2x$, as this will appear when differentiating.



If the form which you would expect the particular integral to take is the same as part of the complementary function, this is a problem, since this means that any particular integral satisfies the homogeneous equation and therefore does not satisfy the whole equation.

For example, you would expect the differential equation $\frac{dy}{dx} - 4\frac{dy}{dx} + 4 = e^{2x}$ to have a particular integral of the form ae^{2x}, but this will not work since it satisfies the equation $\frac{dy}{dx} - 4\frac{dy}{dx} + 4 = 0$,. In such cases, a particular integral of the form axe^{2x} is usually used. However, in this case, this form is also the same as part of the complementary function (since the auxiliary equation has a repeated root) so instead a particular integral of the form ax^2e^{2x} is used.

This situation can also arise with trig functions, as shown in the example below.

Example 2
Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \cos 2x + \sin 2x .$$

Solution
The auxiliary equation is $m^2 + 4 = 0$

$$m = \pm 2i$$

Complementary function is $y = A\cos 2x + B\sin 2x$

The particular integral is of the form $y = x(a\cos 2x + b\sin 2x)$

Since the general solution involves sin $2x$ and cos $2x$, this form of the particular integral is used.

$$\frac{dy}{dx} = a\cos 2x + b\sin 2x + x(-2a\sin 2x + 2b\cos 2x)$$

$$\frac{d^2y}{dx^2} = -2a\sin 2x + 2b\cos 2x + (-2a\sin 2x + 2b\cos 2x) + x(-4a\cos 2x - 4b\sin 2x)$$

$$= -4a\sin 2x + 4b\cos 2x - 4x(a\cos 2x + b\sin 2x)$$

Substituting into the differential equation:

$$-4a\sin 2x + 4b\cos 2x - 4x(a\cos 2x + b\sin 2x) + 4x(a\cos 2x + b\sin 2x) = \cos 2x + \sin 2x$$

$$-4a\sin 2x + 4b\cos 2x = \cos 2x + \sin 2x$$

Equating coefficients of sin x: $-4a = 1 \Rightarrow a = -\frac{1}{4}$

Equating coefficients of cos x: $4b = 1 \Rightarrow b = \frac{1}{4}$

Particular integral is $y = x\left(\frac{1}{4}\sin 2x - \frac{1}{4}\cos 2x\right)$

General solution is $y = A\cos 2x + B\sin 2x + x\left(\frac{1}{4}\sin 2x - \frac{1}{4}\cos 2x\right)$.

Finding general and particular solutions

Solving a differential equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ is quite a lengthy process. However, although it is easy to make careless errors, the steps involved are all quite straightforward.

Here is a summary of the process.

- Write down and solve the quadratic auxiliary equation.
- Use the solutions of the auxiliary equation to write down the complementary function in the appropriate form, with unknown constants A and B.
- Look at the function $f(x)$ on the right-hand side of the differential equation. Check whether it is of the same form as any part of the complementary

function, and then write down the appropriate form of the particular integral, which will involve one, two or three unknown constants.
- Differentiate the particular integral twice, and substitute into the original differential equation. Then equate coefficients to find the values of the unknown constants in the particular integral.
- Write down the general solution of the differential equation by adding the particular integral to the complementary function.
- If a particular solution is required, substitute the given conditions to find the values of the unknown constants A and B. (This may involve solving a pair of simultaneous equations).

Here is a further example.

Example 3
Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \sin x$$

in the case where $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

Solution
The auxiliary equation is:

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1$$

The complementary function is:

$$y = (A + Bx)e^{-x}$$

The particular integral is $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$
$$\frac{d^2y}{dx^2} = -a\sin x - b\cos x$$

Substituting into the differential equation:

$$-a\sin x - b\cos x + 2(a\cos x - b\sin x) + a\sin x + b\cos x = \sin x$$
$$(-a - 2b + a)\sin x + (-b + 2a + b)\cos x = \sin x$$
$$-2b\sin x + 2a\cos x = \sin x$$

Equating coefficients of $\cos x$: $a = 0$
Equating coefficients of $\sin x$: $b = -\frac{1}{2}$

The particular integral is $y = -\frac{1}{2}\cos x$

The general solution is:

$$y = (A + Bx)e^{-x} - \frac{1}{2}\cos x$$

When $x = 0$, $y = 0 \Rightarrow 0 = A - \frac{1}{2} \Rightarrow A = \frac{1}{2}$

$$\frac{dy}{dx} = Be^{-x} - (A + Bx)e^{-x} + \tfrac{1}{2}\sin x$$

When $x = 0$, $\frac{dy}{dx} = 0 \Rightarrow 0 = B - A \Rightarrow B = \frac{1}{2}$

The particular solution is $y = \frac{1}{2}(1 + x)e^{-x} - \frac{1}{2}\cos x$