Section 1: Homogeneous differential equations

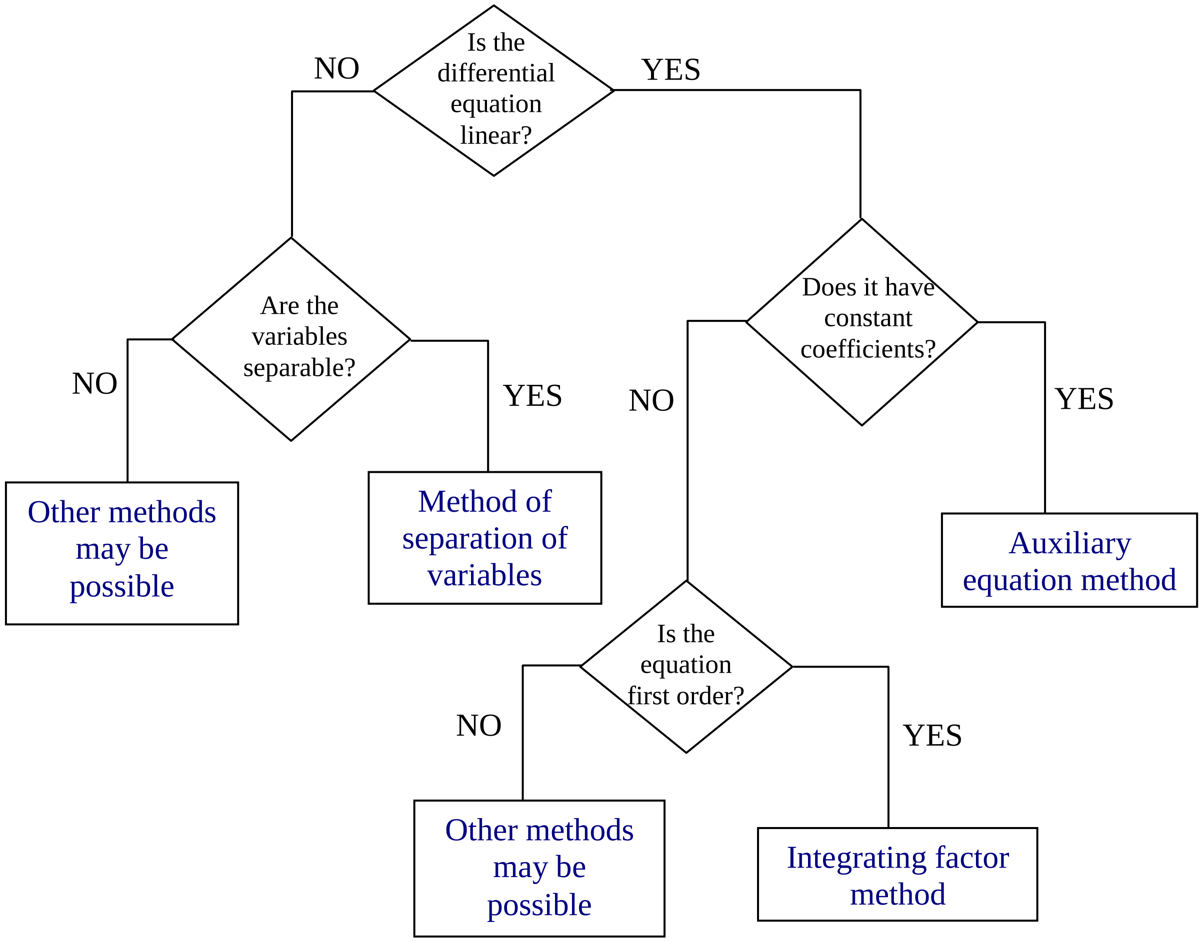
Notes and Examples

These notes contain subsections on

Classification of differential equations

It is important to understand the classification of differential equations because the different methods of solution apply to particular types of equation. For example, you have already learnt two methods which are applicable only to first order linear differential equations. In this topic you will be looking at a method (the auxiliary equation method) which can be applied to linear differential equations of any order, but only if it has constant coefficients.

Uses of the different methods can be illustrated using the following flow chart.



Edexcel FM Second order DEs 1 Notes & Examples

Some differential equations may be solved by more than one method.

Many differential equations cannot be solved by analytical methods. There are numerical methods which can be used to obtain solutions to any degree of accuracy required.

The auxiliary equation method

The auxiliary equation method for solving second order differential equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$$

can be summarised as follows:

- Write down the auxiliary equation $am^2+bm+c=0$
- Solve the equation to obtain either two distinct real roots, one repeated root, or two complex roots (which may be pure imaginary)
- Write down the general solution in the appropriate form, as given in the table below.
  Notice that there are two arbitrary constants in the general solution for a second order differential equation.

Roots of auxiliary equation	**Form of general solution**
Two distinct real roots α and β	$y=A\mathrm{e}^{\alpha x}+B\mathrm{e}^{\beta x}$
One repeated root m	$y=(A+Bx)\mathrm{e}^{mx}$
Pure imaginary roots $\pm n\mathrm{i}$	$y=A\cos nx+B\sin nx$
Complex roots $p\pm q\mathrm{i}$	$y=\mathrm{e}^{px}(A\cos qx+B\sin qx)$

- If initial conditions or boundary conditions are given, use these to find the particular solution.
  Note that two initial or boundary conditions are needed to find the values of the two arbitrary constants.

Example 1
Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=0$$

given that $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=1$ when $x=0$.

Solution

Auxiliary equation: $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2$ or $m = 3$

General solution is $y = Ae^{2x} + Be^{3x}$

When $x = 0, y = 1 \Rightarrow A + B = 1$

$\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x}$

When $x = 0, \frac{dy}{dx} = 1 \Rightarrow 2A + 3B = 1$

Solving the simultaneous equations gives $A = 2, B = -1$

Particular solution is $y = 2e^{2x} - e^{3x}$

Other orders of differential equations

The same principle can be applied to differential equations of any order, so long as it has constant coefficients. The auxiliary equation is of the same order as the differential equation. So a first order differential equation has a linear auxiliary equation, with just one root. A third order differential equation has a cubic auxiliary equation, which has three roots (two of which could be complex), and so on.

The example below shows how a first order differential equation with complex coefficients may be solved by any one of three methods.

Example 2

Find the general solution of the differential equation

$$2\frac{dy}{dx} + 3y = 0.$$

Solution 1: separation of variables

$2\frac{dy}{dx} + 3y = 0$

$\int \frac{1}{y} dy = \int -\frac{3}{2} dx$

$\ln y = -\frac{3}{2}x + c$

$y = e^{-\frac{3}{2}x + c} = Ae^{-\frac{3}{2}x}$

Solution 2: using an integrating factor

$\frac{dy}{dx} + \frac{3}{2}y = 0$

Integrating factor $= e^{\int \frac{3}{2} dx} = e^{\frac{3}{2}x}$

$$e^{\frac{3}{2}x}\frac{dy}{dx}+\frac{3}{2}e^{\frac{3}{2}x}y=0$$

$$\frac{d}{dx}\left(ye^{\frac{3}{2}x}\right)=0$$

$$ye^{\frac{3}{2}x}=A$$

$$y=Ae^{-\frac{3}{2}x}$$

Solution 3: the auxiliary equation method

The auxiliary equation is: $2m+3=0$

$m=-\frac{3}{2}$

The general solution is $y=Ae^{-\frac{3}{2}x}$

The example above demonstrates that the auxiliary equation method is the quickest and easiest for this type of equation.

Simple harmonic motion

A differential equation of the form $\frac{d^2x}{dt^2}+\omega^2x=0$ describes a particular type of motion called simple harmonic motion. The equation can be solved using the auxiliary equation method, as shown in Example 3.

Example 3

Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2}+9x=0$$

that satisfies the initial conditions $x = 1$ and $\frac{dx}{dt} = 3$ when $t = 0$.

Sketch the graph of this particular solution.

Solution

The auxiliary equation is $\lambda^2+9=0$.

This has roots $\pm 3i$, giving the general solution

$$x=A\sin 3t+B\cos 3t$$

When $t = 0$, $x = 1 \Rightarrow B = 1$

Differentiating:

$$\frac{dx}{dt}=3A\cos 3t-3\sin 3t$$

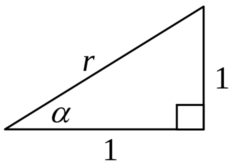
When $t = 0$, $\frac{dx}{dt} = 3 \Rightarrow 3A = 3 \Rightarrow A = 1$

The particular solution is

$$x=\sin 3t+\cos 3t$$

To sketch the graph of the particular solution, it is best to convert the expression $\sin 3t + \cos 3t$ to the form $r\sin(3t+\alpha)$. (This technique is covered in A level Mathematics.)

$$r\cos\alpha = 1$$
$$r\sin\alpha = 1$$
$$r = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

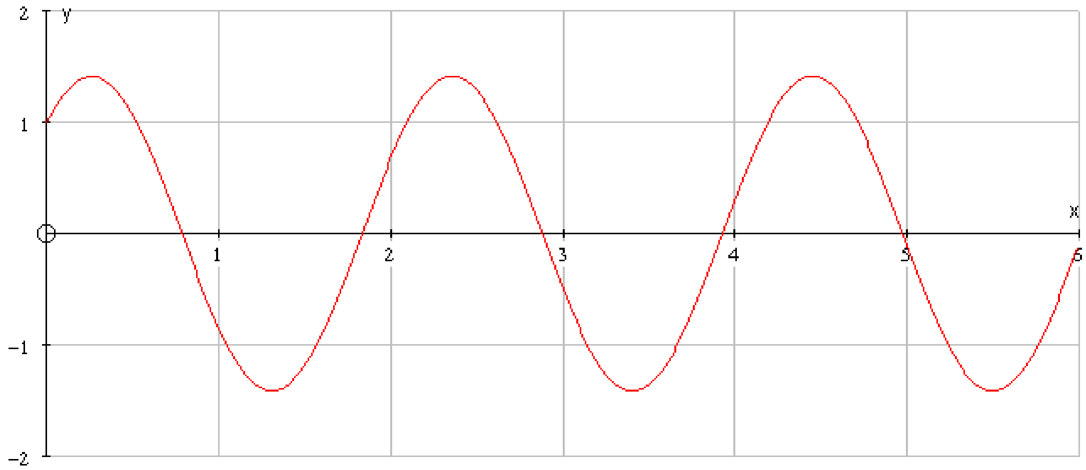


The particular solution can therefore be written as

$$x = \sqrt{2}\sin\left(3t + \frac{\pi}{4}\right)$$

The graph is therefore oscillatory with amplitude $\sqrt{2}$ and period $\frac{2\pi}{3}$.

The initial condition $x = 1$ when $t = 0$ is also helpful in sketching the graph.



All simple harmonic motion is of this form. The solution to the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$ can always be written in the form $x = A\cos\omega t + B\sin\omega t$ or $x = A\sin(\omega t + \varepsilon)$.

Damped oscillations

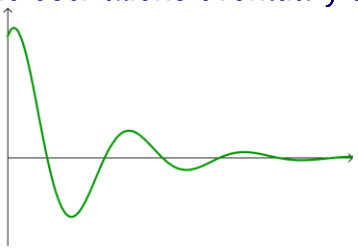
Simple harmonic motion is not always a good model for real-life oscillations, since in most situations the oscillations will eventually die away. This is called damping.

In a linearly damped system, there is a force opposing the motion which is proportional to the speed of the object. This introduces a term in $\frac{\mathrm{d}x}{\mathrm{d}t}$ into the model.

If the auxiliary equation has complex roots $p + qi$, the solution to the differential equation is of the form $x = e^{pt}(A\cos qt + B\sin qt)$.

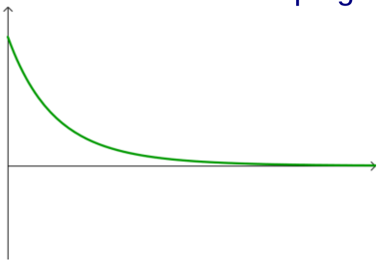
If p is positive, this will result in oscillations with increasing amplitude – so this is not damped motion. Therefore, for damped motion, the value of p must be negative, which means that the coefficient of $\frac{\mathrm{d}x}{\mathrm{d}t}$ in the differential equation must be positive.

- If p is negative, this will result in oscillations with decreasing amplitude, so that the oscillations eventually die away. This is called underdamping

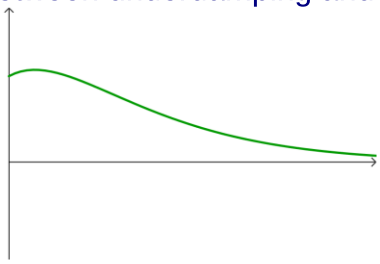


If the auxiliary equation has real roots which are negative (distinct or repeated), these will also result in motion that dies away.

- In the case of real distinct roots which are negative, there are no oscillations. This is called overdamping.



- In the case of a real repeated root which is negative, this is the borderline between underdamping and overdamping.



Roots of auxiliary equation	Form of general solution	Damping
Two distinct real roots α and β	$y = Ae^{\alpha x} + Be^{\beta x}$	If α and β are both negative, the result is overdamping
One repeated root m	$y = (A + Bx)e^{mx}$	If m is negative, the result is critical damping
Pure imaginary roots $\pm n\mathrm{i}$	$y = A\cos nx + B\sin nx$	Simple harmonic motion
Complex roots $p \pm q\mathrm{i}$	$y = e^{px}(A\cos qx + B\sin qx)$	If p is negative, the result is underdamping

You may find it easier to understand the physical significance of the different types of damping by considering a system such as a spring-mass oscillator with a damping device. Imagine pulling down the mass and releasing it. If damping effect is small, the mass will overshoot the equilibrium position and perform oscillations which will gradually decrease – this is underdamping. For a slightly larger damping effect the oscillations will decrease more rapidly. In a particular situation, the mass will not overshoot the equilibrium position and there will be no oscillations. This is the critical damping situation. For an even larger damping effect, the mass will still not overshoot the equilibrium position, but it will take longer to reach it – this is overdamping.