Section 2: The area of a sector

Notes and Examples

These notes contain subsections on

- The area of a sector
- Finding the gradient of a polar curve
- Tangents parallel and perpendicular to the initial line

The area of a sector

When you work in Cartesian coordinates, you find the area under a curve by integrating.

$$\text{Area} = \int_a^b y\,\mathrm{d}x$$

It is very tempting to apply the same idea to polar coordinates, and attempt to integrate r with respect to θ. However, the correct formula is

$$\text{Area} = \int_a^b \tfrac{1}{2}r^2\,\mathrm{d}\theta$$

Whereas in Cartesian coordinates the area under a curve is divided into small rectangles, in polar coordinates the area is divided into small sectors. It is this difference which results in the different expression to be integrated.

When finding the area of a sector, you will almost always need to integrate an expression which involves trig functions. It is, therefore, essential to make sure that you can integrate trig functions confidently. One particular technique which is often needed is the use of the double angle identity to integrate $\cos^2\theta$ and $\sin^2\theta$, as follows:

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \Rightarrow \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

and

$$\cos 2\theta = 1 - 2\sin^2\theta \quad \Rightarrow \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

Here is an example where the double angle identity is also used.

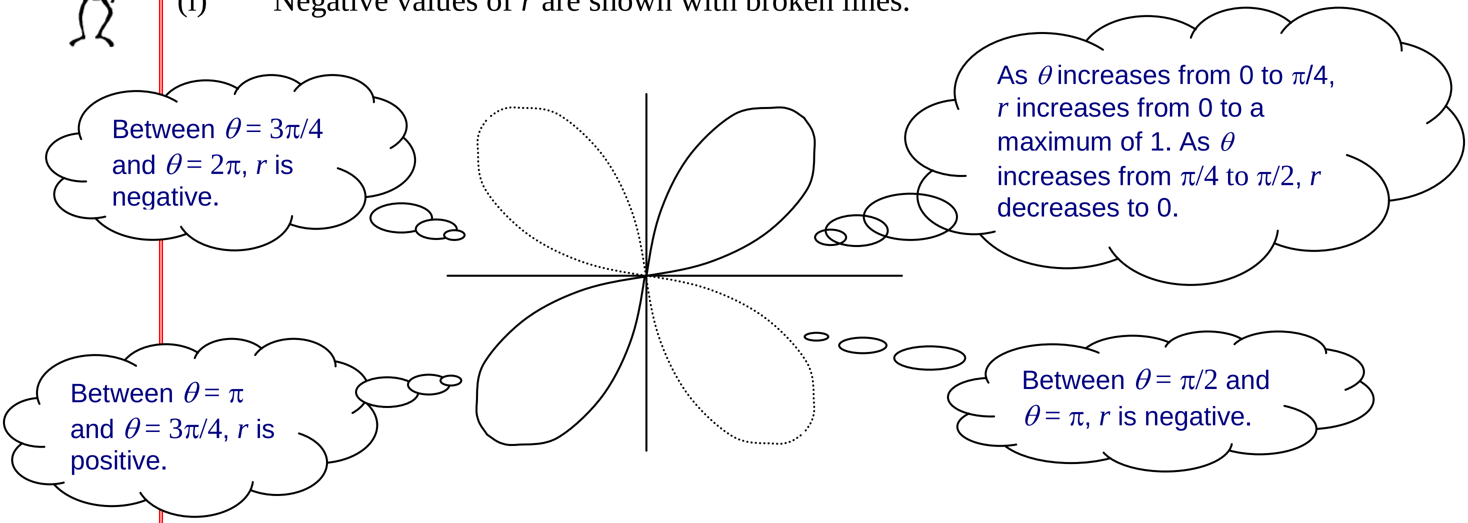
Example 1

(i) Sketch the curve $r = \sin 2\theta$.

(ii) Find the area enclosed by one loop of the curve.

Solution

(i) Negative values of r are shown with broken lines.



(ii) One loop is the part of the curve for $0 \le \theta \le \dfrac{\pi}{2}$.

$$
\begin{aligned}
\text{Area of one loop} &= \int_0^{\pi/2} \tfrac{1}{2} r^2 \, d\theta \\
&= \tfrac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta \\
&= \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4\theta)\, d\theta \\
&= \tfrac{1}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} \\
&= \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) \\
&= \tfrac{1}{8}\pi
\end{aligned}
$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ $\Rightarrow \sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$

Finding the gradient of a polar curve

As with finding the area of a sector, it is tempting to try to find the gradient of a polar curve by differentiating r with respect to θ. This, however, gives you the rate of change of r with respect to θ, which is not the same as the gradient of the polar curve. The gradient of the polar curve is still given by $\dfrac{dy}{dx}$. To find the gradient, it is therefore necessary to convert to Cartesian form before differentiating. Because the Cartesian forms of polar equations are usually quite complicated, it is usually necessary to use implicit differentiation.

Example 2

A curve has polar equation $r^2 = \sin 2\theta$.

(i) Express the equation of the curve in Cartesian form.

(ii) Differentiate this Cartesian equation to find a relationship between x, y and $\dfrac{dy}{dx}$.

(iii) Find the gradient of the curve when $\theta = \dfrac{\pi}{4}$.

Solution

(i) $r^2 = \sin 2\theta$

$r^2 = 2\sin\theta\cos\theta$

$r^4 = 2r\sin\theta \times r\cos\theta$

$(x^2 + y^2)^2 = 2xy$

$x^4 + 2x^2y^2 + y^4 = 2xy$

(ii) Differentiating implicitly:

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

Differentiating x^4

Differentiating $2x^2y^2$ using the product rule

Differentiating y^4

Differentiating $2xy$ using the product rule

(iii) When $\theta = \pi/4$, $r^2 = \sin(\pi/2) = 1 \Rightarrow r = 1$

$x = r\cos\theta = \dfrac{1}{\sqrt{2}}$

$y = r\sin\theta = \dfrac{1}{\sqrt{2}}$

Substituting into the expression in (ii) gives:

$$\frac{4}{2\sqrt{2}} + \frac{4}{2\sqrt{2}} + \frac{4}{2\sqrt{2}}\frac{dy}{dx} + \frac{4}{2\sqrt{2}}\frac{dy}{dx} = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\frac{dy}{dx}$$

$$4 + 4\frac{dy}{dx} = 2 + 2\frac{dy}{dx}$$

$$\frac{dy}{dx} = -1$$

The gradient of the curve when $\theta = \pi/4$ is -1.

Tangents parallel and perpendicular to the initial line

It can sometimes be useful to find the maximum values of x and y on a polar curve. These correspond to the equations of the tangents perpendicular and parallel to the initial line.

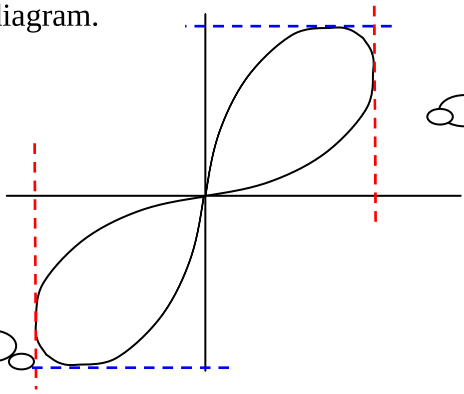
Example 3

A curve has polar equation $r = \sin 2\theta$.

Find the equations of the tangents to the curve parallel and perpendicular to the initial line.

Solution

It is useful to draw a diagram.



The red lines are the tangents perpendicular to the initial line. They occur where $\frac{dx}{d\theta}=0$.

The blue lines are the tangents parallel to the initial line. They occur where $\frac{dy}{d\theta}=0$.

$$x = r\cos\theta = \cos\theta\sin 2\theta$$

$$\frac{dx}{d\theta} = -\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta$$

$$\frac{dx}{d\theta} = 0 \quad \Rightarrow -\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta = 0$$

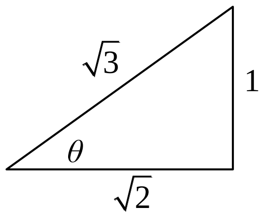
$$\Rightarrow \tan\theta\tan 2\theta = 2$$

$$\Rightarrow \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = 2$$

$$\Rightarrow \cancel{2}\tan^2\theta = \cancel{2}(1-\tan^2\theta)$$

$$\Rightarrow 2\tan^2\theta = 1$$

$$\Rightarrow \tan\theta = \pm\frac{1}{\sqrt{2}}$$



$$\begin{aligned} x &= \cos\theta\sin 2\theta \\ &= \cos\theta \times 2\sin\theta\cos\theta \\ &= 2\sin\theta\cos^2\theta \end{aligned}$$

For $\tan\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{3}}$ and $\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$, so $x = 2\times\frac{1}{\sqrt{3}}\times\frac{2}{3} = \frac{4}{3\sqrt{3}}$.

By symmetry, the two tangents perpendicular to the initial line are $x = \pm\frac{4}{3\sqrt{3}}$.

From the symmetry of the graph, the two tangents parallel to the initial line are $y = \pm\frac{4}{3\sqrt{3}}$.