Section 2: The area of a sector

Section test

1. A spiral has polar equation $r = \dfrac{a\theta}{\pi}$.
   Find the area bounded by the spiral from $\theta = 0$ to $\theta = \pi$ and the initial line.
   Find the area bounded by the spiral from $\theta = 0$ to $\theta = 2\pi$ and the initial line.

2. Find the area of the cardioid $r = 2(1+\cos\theta)$.

3. Find the area of one loop of the curve $r = a\sin 4\theta$.

4. The area bounded by the equiangular spiral $r = ae^{\theta}$ from $\theta = 0$ to $\theta = 2\pi$ and the initial line is

(a) $\frac{1}{4}a^2(e^{2\pi}-1)$ (b) $\frac{1}{2}a^2(e^{4\pi}-1)$

(c) $\frac{1}{4}a^2(e^{4\pi}-1)$ (d) $\frac{1}{2}a^2(e^{2\pi}-1)$

5. A limaçon has equation $r = 1+\sqrt{2}\cos\theta$.
   Find the area of the outer loop of the limaçon.
   Find the area of the inner loop of the limaçon.

6. A curve has polar equation $r^2 = a^2\sin 2\theta$.
   The area of the smallest possible square which encloses the whole curve is

(a) $\dfrac{3}{2}a^2$ (b) $\dfrac{3\sqrt{3}}{8}a^2$

(c) $\dfrac{3\sqrt{3}}{2}a^2$ (d) $\dfrac{3}{8}a^2$

7. A curve has polar equation $r = \cos 3\theta$.
   The Cartesian equation of this curve is $x^4 + 2x^2y^2 + y^4 = x^3 - 3xy^2$.
   Find the area enclosed by one loop of the curve.
   Find the gradient of the curve at the point where $\theta = \frac{2}{3}\pi$.

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Solutions to section test

1. $\text{Area} = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} \frac{a^2\theta^2}{2\pi^2} d\theta$

$$= \frac{a^2}{2\pi^2}\left[\frac{\theta^3}{3}\right]_0^{\pi}$$

$$= \frac{a^2}{2\pi^2} \times \frac{\pi^3}{3}$$

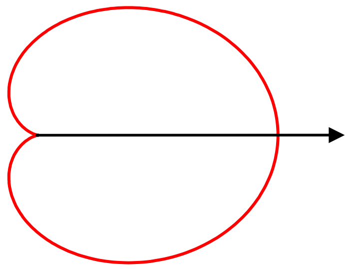
$$= \frac{1}{6}\pi a^2$$

$$\text{Area} = \frac{a^2}{2\pi^2}\left[\frac{\theta^3}{3}\right]_0^{2\pi}$$

$$= \frac{a^2}{2\pi^2} \times \frac{8\pi^3}{3}$$

$$= \frac{4}{3}\pi a^2$$

2.



$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} (2(1+\cos\theta))^2 d\theta$

$$= 2\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 2\int_0^{2\pi} (1 + 2\cos\theta + \tfrac{1}{2}(\cos 2\theta + 1))\, d\theta$$

$$= 2\int_0^{2\pi} (\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta)\, d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= 2 \times \tfrac{3}{2} \times 2\pi$$

$$= 6\pi$$

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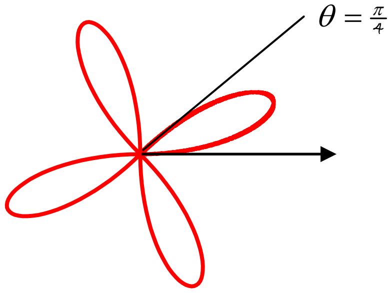
3.



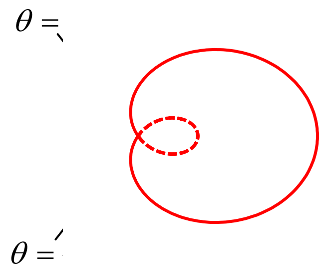
$r = 0$ when $\sin 4\theta = 0 \Rightarrow 4\theta = 0, \pi$

$\Rightarrow \theta = 0, \frac{\pi}{4}$

One loop is therefore between $\theta = 0$ and $\theta = \frac{\pi}{4}$

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2}\int_0^{\pi/4} a^2 \sin^2 4\theta \, d\theta \\
&= \tfrac{1}{2} a^2 \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 8\theta) \, d\theta \\
&= \tfrac{1}{4} a^2 \left[\theta - \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/4} \\
&= \tfrac{1}{4} a^2 \times \tfrac{\pi}{4} \\
&= \tfrac{1}{16} \pi a^2
\end{aligned}$$

4. $$\begin{aligned}
\text{Area} &= \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2}\int_0^{2\pi} a^2 e^{2\theta} \, d\theta \\
&= \tfrac{1}{2} a^2 \left[\tfrac{1}{2} e^{2\theta}\right]_0^{2\pi} \\
&= \tfrac{1}{4} a^2 \left(e^{4\pi} - 1\right)
\end{aligned}$$



5. When $r = 0$, $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

Area is twice the area between $\theta = 0$ and $\theta = \frac{3\pi}{4}$

$$\begin{aligned}
\text{Area} &= 2\int_0^{3\pi/4} \tfrac{1}{2} r^2 \, d\theta = \int_0^{3\pi/4} (1 + \sqrt{2}\cos\theta)^2 \, d\theta \\
&= \int_0^{3\pi/4} (1 + 2\sqrt{2}\cos\theta + 2\cos^2\theta) \, d\theta \\
&= \int_0^{3\pi/4} (1 + 2\sqrt{2}\cos\theta + \cos 2\theta + 1) \, d\theta \\
&= \left[2\theta + 2\sqrt{2}\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{3\pi/4} \\
&= \tfrac{3\pi}{2} + 2\sqrt{2} \times \tfrac{1}{\sqrt{2}} - \tfrac{1}{2} \\
&= \tfrac{3}{2}\pi + \tfrac{3}{2}
\end{aligned}$$

Area is twice the area between $\theta = \frac{3\pi}{4}$ and $\theta = \pi$.

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$$\text{Area} = 2\int_{3\pi/4}^{\pi} \tfrac{1}{2}r^2 d\theta = \int_{3\pi/4}^{\pi} (1+\sqrt{2}\cos\theta)^2 d\theta$$

$$= \left[2\theta + 2\sqrt{2}\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_{3\pi/4}^{\pi}$$

$$= 2\pi + 0 + 0 - \left(\tfrac{3\pi}{2} + 2\sqrt{2}\times\tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\right)$$

$$= \tfrac{1}{2}\pi - \tfrac{3}{2}$$

using integration from above

6. $y = r\sin\theta = a\sqrt{\sin 2\theta}\sin\theta$

$$\frac{dy}{d\theta} = a(\sqrt{\sin 2\theta}\cos\theta + \tfrac{1}{2}\times 2\cos 2\theta(\sin 2\theta)^{-1/2}\sin\theta)$$

$$= a\left(\cos\theta\sqrt{\sin 2\theta} + \frac{\cos 2\theta\sin\theta}{\sqrt{\sin 2\theta}}\right)$$

$$\frac{dy}{d\theta} = 0 \quad \Rightarrow \cos\theta\sqrt{\sin 2\theta} = -\frac{\cos 2\theta\sin\theta}{\sqrt{\sin 2\theta}}$$

$$\Rightarrow \cos\theta\sin 2\theta + \cos 2\theta\sin\theta = 0$$

$$\Rightarrow \sin 3\theta = 0$$

$$\Rightarrow 3\theta = 0 \text{ or } \pm\pi$$

$$\Rightarrow \theta = 0 \text{ or } \pm\frac{\pi}{3}$$

From the graph, $\theta \neq 0$, so the tangent parallel to the initial line in the first quadrant is where $\theta = \frac{\pi}{3}$.

The tangent is therefore $y = a\sqrt{\sin\frac{2\pi}{3}}\sin\frac{\pi}{3}$

By symmetry the tangent perpendicular to the initial line in the first quadrant is $x = a\sqrt{\sin\frac{2\pi}{3}}\sin\frac{\pi}{3}$.

The area of the square is therefore $\left(2a\sqrt{\sin\frac{2\pi}{3}}\sin\frac{\pi}{3}\right)^2$

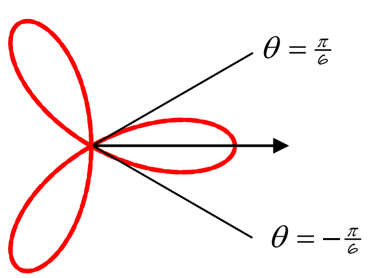
$$= 4a^2\sin\frac{2\pi}{3}\sin^2\frac{\pi}{3}$$

$$= 4a^2\times\frac{\sqrt{3}}{2}\times\frac{3}{4} = \frac{3\sqrt{3}}{2}a^2$$

7. $r = 0$ when $\cos 3\theta = 0 \Rightarrow 3\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

$$\Rightarrow \theta = -\tfrac{\pi}{6}, \tfrac{\pi}{6}$$

One loop is therefore between $\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$



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The area is twice the area between $\theta = 0$ and $\theta = \frac{\pi}{6}$.

$$\text{Area } = 2\int_0^{\pi/6} \tfrac{1}{2} r^2 \, d\theta = \int_0^{\pi/6} \cos^2 3\theta \, d\theta$$
$$= \int_0^{\pi/6} \tfrac{1}{2}(\cos 6\theta + 1) \, d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{1}{6}\sin 6\theta + \theta\right]_0^{\pi/6}$$
$$= \tfrac{1}{12}\pi$$

When $\theta = \frac{2}{3}\pi,\ r = \cos 2\pi = 1$

$$x = r\cos\theta = \cos\tfrac{2}{3}\pi = -\tfrac{1}{2}$$
$$y = r\sin\theta = \sin\tfrac{2}{3}\pi = \tfrac{1}{2}\sqrt{3}$$

$$x^4 + 2x^2y^2 + y^4 = x^3 - 3xy^2$$

Differentiating implicitly:

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 3x^2 - 3y^2 - 6xy\frac{dy}{dx}$$
$$(4x^2y + 4y^3 + 6xy)\frac{dy}{dx} = 3x^2 - 3y^2 - 4x^3 - 4xy^2$$

When $x = -\frac{1}{2}$ and $y = \frac{1}{2}\sqrt{3}$

$$\left(\tfrac{1}{2}\sqrt{3} + \tfrac{3}{2}\sqrt{3} - \tfrac{3}{2}\sqrt{3}\right)\frac{dy}{dx} = \tfrac{3}{4} - \tfrac{9}{4} + \tfrac{1}{2} + \tfrac{3}{2}$$
$$\tfrac{1}{2}\sqrt{3}\frac{dy}{dx} = \tfrac{1}{2}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$$