Edexcel Further Maths Polar coordinates

Section 1: Polar coordinates and curves

Notes and Examples

These notes contain subsections on:
- Polar coordinates
- Curve sketching
- The polar equation of a curve

Polar coordinates

You probably first met the idea of coordinates a long time ago! You will have been familiar with the (x, y) system of coordinates (known as the Cartesian system), for many years, initially in two dimensions, later in three. You know that in one dimension only one number is needed to specify a point uniquely, in two dimensions two numbers are required, and in three dimensions three numbers are required.

However, it may not have been obvious to you in the past that the two numbers required to specify a point in two dimensions need not be given in relation to two perpendicular axes. Polar coordinates give an alternative method of describing the location of a point in the two-dimensional plane.

The following examples show how to convert between Cartesian and polar coordinates.

Example 1
The points A, B and C have Cartesian coordinates (3, 4), (-1, 3) and (-2, -2) respectively. Find the polar coordinates of A, B and C.

Solution

For A $\quad r=\sqrt{x^2+y^2}=\sqrt{25}=5$

$$\tan\theta=\frac{y}{x}=\frac{4}{3}$$

$$\Rightarrow \theta=0.93 \text{ or } -2.21$$

As A lies in the 1st quadrant, A has polar coordinates (5, 0.93)

For B $\quad r=\sqrt{x^2+y^2}=\sqrt{10}=3.16$

$$\tan\theta=\frac{y}{x}=-3$$

$$\Rightarrow \theta=-1.25 \text{ or } 1.89$$

As B lies in the 2nd quadrant, B has polar coordinates (3.16, 1.89)

For C $\quad r = \sqrt{x^2 + y^2} = \sqrt{8} = 2.83$

$$\tan\theta = \frac{y}{x} = 1$$

$$\Rightarrow \theta = \frac{\pi}{4} \text{ or } -\frac{3\pi}{4}$$

As C lies in the 3rd quadrant, C has polar coordinates (2.83, $-\frac{3\pi}{4}$)

Notice from this example:

- The equation $\tan\theta = \frac{y}{x}$ always gives two solutions in the range $-\pi < \theta \le \pi$. You must select the correct one by considering in which quadrant the point lies.
- Angles are normally measured in radians when using polar coordinates. If the angle can be expressed as a multiple of π, as for point C in Example 1, then always do so unless you are instructed otherwise.

Example 2

The points X, Y and Z have polar coordinates $\left(4, \frac{\pi}{6}\right)$, $\left(3, -\frac{2\pi}{3}\right)$ and $\left(-1, \frac{3\pi}{5}\right)$ respectively. Find the Cartesian coordinates of X, Y and Z.

Solution

For X $\quad x = r\cos\theta = 4\cos\frac{\pi}{6} = 2\sqrt{3}$

$$y = r\sin\theta = 4\sin\frac{\pi}{6} = 2$$

X has Cartesian coordinates ($2\sqrt{3}$, 2)

For Y $\quad x = r\cos\theta = 3\cos\frac{-2\pi}{3} = -\frac{3}{2}$

$$y = r\sin\theta = 3\sin\frac{-2\pi}{3} = -\frac{3}{2}\sqrt{3}$$

Y has Cartesian coordinates ($-\frac{3}{2}$, $-\frac{3}{2}\sqrt{3}$)

For Z $\quad x = r\cos\theta = -\cos\frac{3\pi}{5} = 0.31$

$$y = r\sin\theta = -\sin\frac{3\pi}{5} = -0.95$$

Z has Cartesian coordinates (0.31, -0.95)

You can see from Example 2 that converting from polar coordinates to Cartesian is easier than converting from Cartesian to polar, as you do not need to decide which angle is the correct one. However, it is a useful check to decide from the polar coordinates in which quadrant the point lies, and ensure that the Cartesian coordinates are consistent with this.

In Example 2 above, for X, $\theta = \frac{\pi}{6}$ which is in the 1st quadrant, for Y, $\theta = -\frac{2\pi}{3}$ which is in the 3rd quadrant and for Z, $\theta = \frac{3\pi}{5}$ which is in the 2nd quadrant, but as r is negative in this case Z must lie in the 4th quadrant.

Curve sketching

The work on sketching polar curves in this chapter is often actually simpler than Cartesian curve sketching. You do not need to solve equations to find points where the curve crosses the axes, nor do you need to use calculus to find turning points. Most of the polar curves you will meet involve simple trigonometric functions, and you simply need to decide whether the function is positive or negative, and increasing or decreasing, at different points.

You can investigate some families of polar curves using the Geogebra resource ***Exploring polar curves***. You can see how the curve develops as the angle increases. The parts of the curve for which r is negative are shown as a broken line.

For the curves $r = 5\cos n\theta$ and $r = 5\sin n\theta$:

- How does the curve change as you increase the value of n?
- What are the differences between odd and even values of n?
- Do you always need to go all the way to 2π to complete the curve?
- How do the sine curves differ from the cosine curves?

For the curves $r = a + b\cos\theta$ and $r = a + b\sin\theta$:

- How does the shape of the curve differ for the cases $a < b$, $a = b$ and $a > b$?
- What happens when b is negative?
- How do the sine curves differ from the cosine curves?

For the curves $r^2 = 25\sin n\theta$:

- How does the curve change as you increase the value of n?
- For what angles is the curve not defined?

Making a table of values and plotting points is one possible approach to sketching polar curves. However, this is time-consuming. As with Cartesian curve sketching, the important thing is to show the main features of the curve. When you sketch a polar curve, you need to start by using the equation to identify important values of θ, where r is zero or takes its maximum or minimum value.

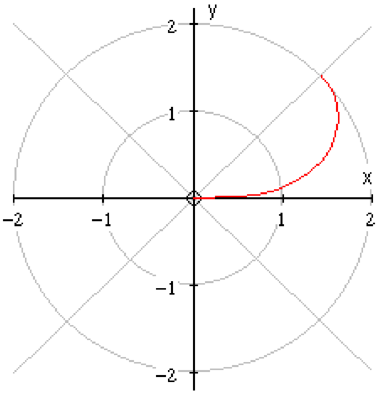
Here is an example using this approach.

Example 3

Sketch the curve $r^2 = 4\sin 2\theta$

Solution

As this is a function of 2θ, angles which are multiples of $\pi/4$ are important points.
Start by considering the positive square root only.

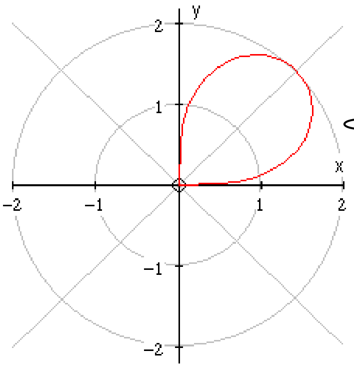


$\theta = \pi/4$

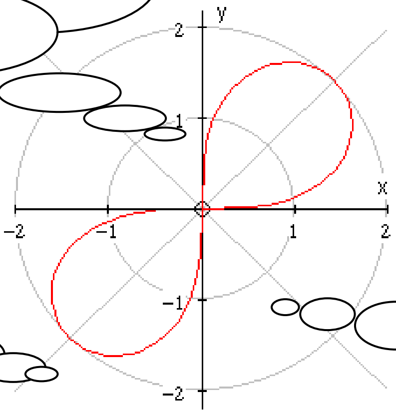
When $\theta = 0$, $r = 0$.
As θ increases to $\pi/4$, r increases to a maximum of 2.

As θ increases from $\pi/4$ to $\pi/2$, r decreases to 0.

$\theta = \pi/4$



As θ increases from $\pi/2$ to π, $\sin 2\theta$ is negative, so there is no part of the graph for these values of θ



As θ increases from π to $3\pi/4$, r increases from 0 to 2, and as θ increases from $3\pi/4$ to $3\pi/2$, r decreases to 0

As θ increases from $3\pi/2$ to 2π, $\sin 2\theta$ is negative, so there is no part of the graph for these values of θ

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Notice that if you consider the negative square root for $0 \le \theta \le \frac{\pi}{2}$, you get the same part of the graph as the positive square root for $\pi \le \theta \le \frac{3\pi}{2}$.

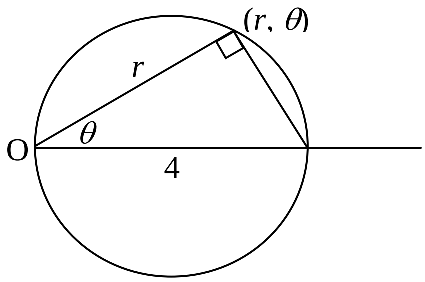
The polar equation of a curve

Learning to work with polar coordinates is a bit like learning a foreign language! When you learn a new language, you decide what you want to say in your own language and then translate it. However, someone who is fluent in a new language begins to think in that language and does not need to mentally translate all the time.

It can be very tempting when dealing with polar coordinates to change everything into the more familiar Cartesian form. However, there are many situations when thinking in polar coordinates makes the mathematics much simpler. Below is an example of a situation like this.

Example 4
Find the polar equation of a circle whose centre has polar coordinates (2, 0), with radius 2.

Solution A (using polar coordinates)



Polar equation is $r = 4\cos\theta$

Solution B (using Cartesian coordinates)

Cartesian equation: $(x-2)^2 + y^2 = 4$

$x^2 - 4x + 4 + y^2 = 4$

$x^2 + y^2 = 4x$

Polar equation: $r^2 = 4r\cos\theta$

$r = 4\cos\theta$

You can see that by thinking geometrically using polar coordinates, you can deduce the polar equation of the curve directly. Using Cartesian coordinates is a valid approach but requires several lines of working.