Section 3: Further integration

Section test

1. Find $\int \frac{1}{\sqrt{7-6x-x^2}}\,\mathrm{d}x$

(a) $\arcsin\left(\frac{x+3}{4}\right)+c$

(b) $\arcsin\left(\frac{x-3}{4}\right)+c$

(c) $\frac{1}{4}\arcsin\left(\frac{x+3}{4}\right)+c$

(d) $\frac{1}{4}\arcsin\left(\frac{x-3}{4}\right)+c$

2. Find $\int_1^2 \frac{1}{3x^2-6x+4}\,\mathrm{d}x$

(a) $\frac{\pi}{\sqrt{3}}$

(b) $\frac{\pi}{3\sqrt{3}}$

(c) $\frac{\pi}{3}$

(d) $\pi\sqrt{3}$

3. Which one of the following can be integrated to obtain an arctangent function?

(a) $\int \frac{1}{4x^2-12x+3}\,\mathrm{d}x$

(b) $\int \frac{1}{x^2-4x+3}\,\mathrm{d}x$

(c) $\int \frac{1}{x^2+6x+8}\,\mathrm{d}x$

(d) $\int \frac{1}{4x^2+4x+3}\,\mathrm{d}x$

4. Which one of the following can be integrated to obtain an arcsine function?

(a) $\int \frac{1}{\sqrt{-3+4x-x^2}}\,\mathrm{d}x$

(b) $\int \frac{1}{\sqrt{5-4x+x^2}}\,\mathrm{d}x$

(c) $\int \frac{1}{\sqrt{-3+4x-4x^2}}\,\mathrm{d}x$

(d) $\int \frac{1}{\sqrt{-10+6x-x^2}}\,\mathrm{d}x$

5. Find $\int \frac{x+2}{\sqrt{4-x^2}}\,\mathrm{d}x$

(a) $2\arcsin\frac{x}{2}-\frac{1}{2}\sqrt{4-x^2}+c$

(b) $2\arcsin\frac{x}{2}+\sqrt{4-x^2}+c$

(c) $2\arcsin\frac{x}{2}-\sqrt{4-x^2}+c$

(d) $2\arcsin\frac{x}{2}+\frac{1}{2}\sqrt{4-x^2}+c$

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6. For $\dfrac{2x+4}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$, find the values of A, B and C.

   Hence find $\displaystyle\int_0^1 \frac{2x+4}{(x-2)(x^2+4)}\,\mathrm{d}x$ in the form $a\ln b$.

7. Find $\displaystyle\int \frac{-13}{(x^2+9)(x-2)}\,\mathrm{d}x$

(a) $\ln\left|\dfrac{\sqrt{x^2+9}}{2(x-2)}\right| + \dfrac{2}{3}\arctan\dfrac{x}{3} + c$

(b) $\ln\left|\dfrac{x^2+9}{2(x-2)}\right| + \dfrac{2}{3}\arctan\dfrac{x}{3} + c$

(c) $\ln\left|\dfrac{x^2+9}{x-2}\right| + \dfrac{2}{3}\arctan\dfrac{x}{3} + c$

(d) $\ln\left|\dfrac{\sqrt{x^2+9}}{x-2}\right| + \dfrac{2}{3}\arctan\dfrac{x}{3} + c$

8. Use a trigonometric substitution to find $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x$.

9. What is the value of $\displaystyle\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,\mathrm{d}x$? Give your answer in the form $a\sqrt{3}$.

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Solutions to section test

1. $7-6x-x^2=16-(x+3)^2$

$$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx=\int \frac{1}{\sqrt{16-(x+3)^2}}\,dx$$
$$=\int \frac{1}{\sqrt{4^2-(x+3)^2}}\,dx$$
$$=\arcsin\left(\frac{x+3}{4}\right)+c$$

2. $3x^2-6x+4=3\left(x^2-2x+\frac{4}{3}\right)$
$$=3\left((x-1)^2+\tfrac{1}{3}\right)$$
$$\int_1^2 \frac{1}{3x^2-6x+4}\,dx=\frac{1}{3}\int_1^2 \frac{1}{(x-1)^2+\frac{1}{3}}\,dx$$
$$=\frac{1}{3}\times\sqrt{3}\left[\arctan\left((x-1)\sqrt{3}\right)\right]_1^2$$
$$=\frac{1}{\sqrt{3}}\left(\arctan\sqrt{3}-\tan^{-1}0\right)$$
$$=\frac{1}{\sqrt{3}}\times\frac{\pi}{3}-0$$
$$=\frac{\pi}{3\sqrt{3}}$$

3. $4x^2+4x+3=4\left(x^2+x+\frac{3}{4}\right)=4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right)$

$x^2-4x+3=(x-2)^2-1$

$x^2+6x+8=(x+3)^3-1$

$4x^2-12x+3=4\left(x^2-3x+\frac{3}{4}\right)=4\left(\left(x-\frac{3}{2}\right)^2-\frac{3}{2}\right)$

$\dfrac{1}{4x^2+4x+3}$ is the only one which can be expressed in the form $\dfrac{A}{a^2+(x+b)^2}$ so is the only one which can be integrated to give an arctan function. All the rest can be expressed in the form $\dfrac{A}{(x+b)^2-a^2}$

4. $-3+4x-x^2=1-(2-x)^2$

$5-4x+x^2=1+(x-2)^2$

$-3+4x-4x^2=4\left(-\frac{3}{4}+x-x^2\right)=4\left(-\frac{1}{2}-\left(x-\frac{1}{2}\right)^2\right)$

$-10+6x-x^2=-1-(x-3)^2$

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$\frac{1}{\sqrt{-3+4x-x^2}}$ is the only one which can be expressed in the form $\frac{A}{\sqrt{a^2-(x+b)^2}}$.

All the others can be expressed in the form $\frac{A}{\sqrt{a^2+(x+b)^2}}$ or $\frac{A}{\sqrt{-a^2-(x+b)^2}}$.

5. $$\int \frac{x+2}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx$$
$$= -\left(4-x^2\right)^{\frac{1}{2}} + 2\arcsin\frac{x}{2} + c$$
$$= 2\arcsin\frac{x}{2} - \sqrt{4-x^2} + c$$

6. $$\frac{2x+4}{(x-2)(x^2+4)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$
$$2x+4 \equiv A\left(x^2+4\right) + (Bx+C)(x-2)$$
When $x=2$: $\quad 8 = 8A$

$A = 1$

By equating the constants: $\quad 4 = 4A - 2C$

$C = 0$

By equating the coefficients of x: $\quad 2 = -2B + C$

$B = -1$

$$\frac{2x+4}{(x-2)(x^2+4)} \equiv \frac{1}{x-2} - \frac{x}{x^2+4}$$

$$\int_0^1 \frac{2x+4}{(x-2)(x^2+4)}\,dx = \int_0^1 \left(\frac{1}{x-2} - \frac{x}{x^2+4}\right)dx$$
$$= \left[\ln|x-2| - \tfrac{1}{2}\ln(x^2+4)\right]_0^1$$
$$= \ln 1 - \tfrac{1}{2}\ln 5 - \ln 2 + \tfrac{1}{2}\ln 4$$
$$= -\tfrac{1}{2}\ln 5 - \ln 2 + \ln 2$$
$$= -\tfrac{1}{2}\ln 5$$

7. $$\frac{-13}{(x^2+9)(x-2)} = \frac{Ax+B}{x^2+9} + \frac{C}{x-2}$$

Write the integral in partial fractions

$$-13 = (Ax+B)(x-2) + C\left(x^2+9\right)$$
Let $x=2$ $\quad -13 = 13C \Rightarrow C = -1$

Let $x=0$ $\quad -13 = -2B + 9C \Rightarrow -2B = -4 \Rightarrow B = 2$

Comparing coefficients of x^2: $\quad A + C = 0 \Rightarrow A = 1$

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$$\int \frac{-13}{(x^2+9)(x-2)}\,dx = \int\left(\frac{x+2}{x^2+9} - \frac{1}{x-2}\right)dx$$

$$= \int \frac{x}{x^2+9}\,dx + \int \frac{2}{x^2+9}\,dx - \int \frac{1}{x-2}\,dx$$

$$= \tfrac{1}{2}\ln\left(x^2+9\right) + \frac{2}{3}\arctan\frac{x}{3} - \ln|x-2| + c$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{x-2}\right| + \frac{2}{3}\arctan\frac{x}{3} + c$$

8. Let $x = \sin u$

$$\frac{dx}{du} = \cos u$$

When $x = 0, u = 0$

When $x = 1, u = \frac{\pi}{2}$

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sin^3 u\sqrt{1-\sin^2 u} \times \cos u\,du$$

using the identity $\sin^2 x = 1 - \cos^2 x$

$$= \int_0^{\pi/2} \sin^3 u\cos^2 u\,du$$

$$= \int_0^{\pi/2} \sin u(1-\cos^2 u)\cos^2 u\,du$$

$$= \int_0^{\pi/2} (\sin u\cos^2 u - \sin u\cos^4 u)\,du$$

$$= \left[-\tfrac{1}{3}\cos^3 u + \tfrac{1}{5}\cos^5 u\right]_0^{\pi/2}$$

$$= 0 - (-\tfrac{1}{3} + \tfrac{1}{5})$$

$$= \tfrac{2}{15}$$

Both these terms can be integrated by inspection since $\sin x$ is the derivative of $\cos x$. Alternatively use the substitution $u = \cos x$

9. Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$

When $x = 0, \theta = 0$

When $x = 1, \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\pi/6} \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}} \times 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \frac{1}{8\cos^3\theta} \times 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \tfrac{1}{4}\sec^2\theta\,d\theta$$

$$= \left[\tfrac{1}{4}\tan\theta\right]_0^{\pi/6}$$

$$= \tfrac{1}{4} \times \frac{1}{\sqrt{3}} = \tfrac{1}{12}\sqrt{3}$$

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