Section 3: Further integration

Notes and Examples

These notes contain subsections on:

- Partial fractions
- Harder integrations
- Trigonometric substitutions

Partial fractions

In A level Mathematics, you learned to express fractions of the form $\frac{px+q}{(ax+b)(cx+d)}$, $\frac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$ and $\frac{px^2+qx+r}{(ax+b)^2(cx+d)}$ in partial fractions: that is, as a sum of simpler fractions. One application of partial fractions is that the resulting fractions are usually easy to integrate.

You will now look at another form of expression that can be written in partial fractions: $\frac{px^2+qx+r}{(ax^2+b)(cx+d)}$. Notice that this involves a quadratic denominator that cannot be factorised. Expressions of this form can be written as $\frac{Ax+B}{(ax^2+b)}+\frac{C}{cx+d}$.

Example 1 shows how this is done.

Example 1

Express $\frac{2x-7}{(x^2+4)(x-1)}$ in partial fractions.

Solution

$$\frac{2x-7}{(x^2+4)(x-1)}=\frac{Ax+B}{x^2+4}+\frac{C}{x-1}$$
$$2x-7=(Ax+B)(x-1)+C(x^2+4)$$

Let $x = 1$: $\quad -5=5C \quad \Rightarrow C=-1$
Let $x = 0$: $\quad -7=-B+4C \quad \Rightarrow -7=-B-4 \quad \Rightarrow B=3$
Equating powers of x^2: $\quad 0=A+C \quad \Rightarrow A=1$

$$\frac{2x-7}{(x^2+4)(x-1)}=\frac{x+3}{x^2+4}-\frac{1}{x-1}$$

Notice that to find the values of A, B, and C, a mixture of substituting numbers and equating coefficients is used. You can do this just by substituting numbers, or just be equating coefficients, if you prefer, but the method shown above is perhaps the most efficient in most cases.

Once an expression of this form has been expressed in partial fractions, it can be integrated. The first term needs to be split into two to give $\frac{x}{x^2+4}+\frac{3}{x^2+4}$. You should be able to see that $\frac{x}{x^2+4}$ can be integrated by inspection (or by substitution) since the numerator is a multiple of the derivative of the denominator, and $\frac{3}{x^2+4}$ is recognisable as an integral which will lead to an arctan function.

Example 2

Find $\int \frac{2x-7}{(x^2+4)(x-1)}\,\mathrm{d}x$.

Solution

$$\int \frac{2x-7}{(x^2+4)(x-1)}\,\mathrm{d}x = \int \frac{x}{x^2+4}\,\mathrm{d}x + \int \frac{3}{x^2+4}\,\mathrm{d}x - \int \frac{1}{x-1}\,\mathrm{d}x$$

$$= \tfrac{1}{2}\ln({}^2+4) + \tfrac{3}{2}\arctan\frac{x}{2} - \ln|x-1| + c$$

$$= \ln\frac{\sqrt{x^2+4}}{|x-1|} + \tfrac{3}{2}\arctan\frac{x}{2} + c$$

Harder integrations

You can use the technique of completing the square to integrate functions of the form $\frac{1}{Ax^2+Bx+C}$ and $\frac{1}{\sqrt{Ax^2+Bx+C}}$.

- **Harder integration using the arctan function**

If the function $\frac{1}{Ax^2+Bx+C}$ can be written in the form $\frac{1}{A}\times\frac{1}{a^2+(x+b)^2}$ then it can be integrated to give an arctan function:

$$\int \frac{1}{a^2+(x+b)^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x+b}{a}\right)+c$$

Notice that, as before, if the coefficient of x^2 is not 1, you must take out a factor first. Many students find completing the square in these circumstances difficult. You may find it easier to deal with this separately before attempting the integration.

Example 3

Find $\int \frac{1}{4x^2+6x+3}\,\mathrm{d}x$.

Solution

First write the denominator in the completed square form:

$$\begin{aligned} 4x^2+6x+3 &= 4\left(x^2+\tfrac{3}{2}x+\tfrac{3}{4}\right) \\ &= 4\left((x^2+\tfrac{3}{2}x+\tfrac{9}{16})-\tfrac{9}{16}+\tfrac{3}{4}\right) \\ &= 4((x+\tfrac{3}{4})^2+\tfrac{3}{16}) \end{aligned}$$

Now

$$\begin{aligned} \int \frac{1}{4x^2+6x+3}\,\mathrm{d}x &= \frac{1}{4}\int \frac{1}{(x+\frac{3}{4})^2+\frac{3}{16}}\,\mathrm{d}x \\ &= \frac{1}{4}\times\frac{4}{\sqrt{3}}\arctan\left(\frac{4}{\sqrt{3}}(x+\tfrac{3}{4})\right)+c \\ &= \frac{1}{\sqrt{3}}\arctan\frac{4x+3}{\sqrt{3}}+c \end{aligned}$$

using the standard integral with $a=\frac{\sqrt{3}}{4}$

- **Harder integration using the arcsin function**

If the function $\frac{1}{\sqrt{Ax^2+Bx+C}}$ can be written in the form $\frac{1}{\sqrt{|A|}}\frac{1}{\sqrt{a^2-(x+b)^2}}$ then it can be integrated to give an arcsin function:

$$\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,\mathrm{d}x = \arcsin\left(\frac{x+b}{a}\right)+c.$$

Again, if the coefficient of x^2 is not 1, you must take out a factor first.

Example 4

Find $\int \frac{1}{\sqrt{5-8x-4x^2}}\,\mathrm{d}x$

Solution

First write the denominator in the completed square form.

$$\begin{aligned} 5-8x-4x^2 &= 4\left(\tfrac{5}{4}-2x-x^2\right) \\ &= 4\left(\tfrac{5}{4}-(x^2+2x)\right) \\ &= 4\left(\tfrac{5}{4}-(x^2+2x+1)+1\right) \\ &= 4\left(\tfrac{9}{4}-(x+1)^2\right) \end{aligned}$$

Now $\int \frac{1}{\sqrt{5-8x-4x^2}}\,dx = \int \frac{1}{2\sqrt{\frac{9}{4}-(x+1)^2}}\,dx$

$= \frac{1}{2}\arcsin\left(\frac{2}{3}(x+1)\right)+c$

using the standard integral with $a=\frac{3}{2}$

Trigonometric substitutions

The standard results $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+c$ and $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+c$ can be derived by using the substitutions $x=a\sin\theta$ and $x=a\tan\theta$ respectively. You don't normally need to carry out these substitutions, because you can simply quote and use the standard integrals. However, other integrals which involve functions of $\sqrt{a^2-x^2}$ or a^2+x^2 can sometimes be found by using these substitutions and applying identities.

Example 5

Find $\int \sqrt{9-x^2}\,dx$.

Solution

Use the substitution $x=3\sin\theta$

$\frac{dx}{d\theta}=3\cos\theta \quad \Rightarrow dx = 3\cos\theta\,d\theta$

$$\begin{aligned}
\int\sqrt{9-x^2}\,dx &= \int\sqrt{9-9\sin^2\theta}\times 3\cos\theta\,d\theta \\
&= \int 3\sqrt{1-\sin^2\theta}\times 3\cos\theta\,d\theta \\
&= 9\int\cos^2\theta\,d\theta \\
&= \tfrac{9}{2}\int(\cos 2\theta+1)\,d\theta \\
&= \tfrac{9}{2}\left(\tfrac{1}{2}\sin 2\theta+\theta\right)+c \\
&= \tfrac{9}{2}\sin\theta\cos\theta+\tfrac{9}{2}\theta+c \\
&= \tfrac{9}{2}\times\tfrac{x}{3}\sqrt{1-\left(\tfrac{x}{3}\right)^2}+\tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right)+c \\
&= \tfrac{3}{2}x\times\tfrac{1}{3}\sqrt{9-x^2}+\tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right)+c \\
&= \tfrac{1}{2}x\sqrt{9-x^2}+\tfrac{9}{2}\arcsin\left(\tfrac{x}{3}\right)+c
\end{aligned}$$