Section 2: Inverse trigonometric functions

Notes and Examples

These notes contain subsections on:

- The inverse trigonometric functions (reminder)
- Differentiation of functions involving inverse trigonometric functions
- Integration using inverse trigonometric functions

The inverse trigonometric functions (reminder)

The inverse trigonometric functions arcsin, arccos and arctan (sometimes written as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$) are introduced in A level Mathematics. Remember that since the functions sin, cos and tan are not one-to-one, they do not have inverses, so the inverse trigonometric functions are the inverses of the trigonometric functions defined for a restricted domain, over which they are one-to-one.

In this section you will learn to differentiate the inverse trig functions and to apply this knowledge to integration.

Differentiation of functions involving inverse trig functions

This section introduces the three standard results:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

You can prove these by using implicit differentiation.

$$y = \arcsin x$$
$$\sin y = x$$
$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

The other two results can be proved in a similar way.

You also need to be able to differentiate other functions involving the inverse trig functions, by using the standard results together with the chain rule, as shown in Example 1 below.

Example 1

Differentiate the following with respect to x.

(i) $\arccos 2x$

(ii) $\arcsin(3-x)$

(iii) $\arctan(x^2+1)$

You have done many differentiations like this in the past, using the chain rule with a variety of functions. You should be able to see that the above example is done in the same way, just with a different function. You may feel confident enough with this kind of work to be able to do it directly, without writing down the substitution. The second two parts of this example are done like this.

Solution

(i) Let $u = 2x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos 2x) = \frac{\mathrm{d}}{\mathrm{d}u}(\arccos u) \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

$$= \frac{-1}{\sqrt{1-u^2}} \times 2$$

$$= \frac{-2}{\sqrt{1-4x^2}}$$

(ii) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin(3-x)\right) = \frac{1}{\sqrt{1-(3-x)^2}} \times -1$$

$$= \frac{-1}{\sqrt{1-(3-x)^2}}$$

The derivative of $3-x$ is -1.

(iii) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan(x^2+1)\right) = \frac{1}{1+(x^2+1)^2} \times 2x$$

$$= \frac{2x}{1+(x^2+1)^2}$$

The derivative of $x^2 + 1$ is $2x$.

Integration using inverse trigonometric functions

In this section you learn to apply the two standard integrals

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + c\,.$$

Note the following points when using these integrals:

- If the numerator is a constant other than 1, then no problem: the integral is simply multiplied by this constant.

- You cannot integrate $\frac{1}{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$ or $\frac{1}{\sqrt{x^2 + a^2}}$ using these standard integrals. However, you will learn to integrate these functions in later work on hyperbolic functions.

- The coefficient of x^2 must be 1 to apply these standard integrals. If it is not, you must take out a factor first, or use a substitution.

- Remember, when taking a factor b out of $\sqrt{a^2 - bx^2}$, you need to take the square root, so that you get $\sqrt{b}\sqrt{\frac{a^2}{b} - x^2}$.

Example 2

Find $\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,\mathrm{d}x$.

Solution

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,\mathrm{d}x = \int \frac{1}{b\sqrt{\frac{a^2}{b^2} - x^2}}\,\mathrm{d}x$$

Taking out a factor b

$$= \frac{1}{b}\int \frac{1}{\sqrt{\frac{a^2}{b^2} - x^2}}\,\mathrm{d}x$$

$$= \frac{1}{b}\arcsin\left(\frac{x}{a/b}\right) + c$$

$$= \frac{1}{b}\arcsin\left(\frac{bx}{a}\right) + c$$

using the standard integral with $\frac{a}{b}$ instead of a