Advanced Mathematics Support Programme®

Edexcel A Level FM Revision Questions: Solutions

Hyperbolic functions; Maclaurin series

Question 1

$$4\,\frac{e^x - e^{-x}}{e^x + e^{-x}} + 13\,\frac{2}{e^x + e^{-x}} = 8$$

$$\Rightarrow \qquad 2e^x - 2e^{-x} + 13 = 4e^x + 4e^{-x}$$

$$\Rightarrow \qquad 0 = 2e^x - 13 + 6e^{-x}$$

$$\Rightarrow \qquad 2(e^x)^2 - 13e^x + 6 = 0$$

$$\Rightarrow \qquad (2e^x - 1)(e^x - 6) = 0$$

$$\Rightarrow \qquad e^x = \frac{1}{2} \quad \Rightarrow x = -\ln(2)$$

$$\Rightarrow \qquad e^x = 6 \quad \Rightarrow x = \ln(6)$$

Question 2

(i) $let\ y = \sinh(x)$

$$\Rightarrow y = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \quad (e^y)^2 - 2x\,e^y - 1 = 0$$

$$\Rightarrow \quad e^y = \frac{4x \pm \sqrt{4x^2 + 4}}{2}$$

$$\Rightarrow \quad e^y = x + \sqrt{x^2 + 1} \quad as\ \ e^y\ is\ always + ve$$

$$\Rightarrow \quad y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\Rightarrow \quad \operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

(ii) Using quotient differentiation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(2+\sinh x).\sinh x - \cosh x \cosh x}{(2+\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh x + \ \sinh^2 x - \cosh^2 x}{(2+\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh x - 1}{(2+\sinh x)^2}$$

$But\ \frac{dy}{dx} = \frac{1}{9} \qquad \Rightarrow \qquad \frac{2\sinh x - 1}{(2+\sinh x)^2} = \frac{1}{9}$

$$18\sinh x - 9 = 4 + 4\sinh x + \sinh^2 x$$

$$\sinh^2 x - 14\sinh x + 13 = 0$$

$$(\sinh x - 1)(\sinh x - 13) = 0$$

$either\ \sinh x = 1 \Rightarrow \quad x = \ln\left(1+\sqrt{2}\right) \Rightarrow y = \frac{\sqrt{2}}{3} \qquad \left\{as\ y = \frac{\sqrt{1+\sinh^2 x}}{2+\sinh x}\right\}$

$or \qquad \sinh x = 13 \Rightarrow \quad x = \ln\left(13+\sqrt{170}\right) \Rightarrow y = \frac{\sqrt{170}}{15}$

Question 3

$$\int_1^2 \frac{1}{\sqrt{9x^2-4}}dx = \frac{1}{3}\int_1^2 \frac{1}{\sqrt{x^2-\left(\frac{2}{3}\right)^2}}dx$$

$standard\ result \int \frac{1}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

$$= \frac{1}{3}\left[\ln\left(x + \sqrt{x^2 - \frac{4}{9}}\right)\right]_1^2$$

$$= \frac{1}{3}\left\{\ln\left(2 - \frac{2\sqrt{2}}{3}\right) - \ln\left(1 + \frac{\sqrt{5}}{3}\right)\right\}$$

$$= \frac{1}{3}\ln\left(\frac{6 - 2\sqrt{2}}{3-\sqrt{5}}\right)$$

Question 4

$$1 + 2\sinh^2 x = 1 + \frac{2\,(e^x - e^{-x})^2}{4}$$

$$= 1 + \frac{2e^{2x} - 4 + 2e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh(2x)$$

<u>Question 5</u>

Using $x = 2\sinh u \quad \Rightarrow \quad dx = 2\cosh u \, du$

$$\int \sqrt{x^2 + 4}\, dx = \int \sqrt{4\sinh^2 u + 4} \times 2\cosh u \, du$$

$$= \int 4\cosh^2 u \, du$$

$$= \int (2\cosh 2u + 2)\, du$$

$$= \sinh 2u + 2u + c$$

$$= 2\sinh u \cosh u + 2u + c$$

$$= \frac{1}{2} x \sqrt{x^2 + 4} + 2\,\text{arsinh}\left(\frac{x}{2}\right) + c$$

(b) $t^2 + 2t + 5 = (t+1)^2 + 4$

$$\int_{-1}^{1} \sqrt{t^2 + 2t + 5}\; dt = \int_{-1}^{1} \sqrt{(t+1)^2 + 4}\; dt$$

$$= \left[\frac{1}{2}(t+1)\sqrt{(t+1)^2 + 4} + 2\,\text{arsinh}\left(\frac{t+1}{2}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\, 2\sqrt{8} + 2\,\text{arsinh}(1) - 2\,\text{arsinh}(0)$$

$$= 2\sqrt{2} + 2\ln\left(1 + \sqrt{2}\right)$$

$$\int_{-1}^{1} \sqrt{t^2 + 2t + 5}\; dt = 2\{\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\}$$

<u>Question 6</u>

$\int \sqrt{4x^2 - 25}\, dx \qquad let\ 4x^2 = 25\cosh^2 u \quad \Rightarrow \quad x = \frac{5}{2}\cosh u \Rightarrow \quad dx = \frac{5}{2}\sinh u\, du$

$$\int \sqrt{4x^2 - 25}\, dx = \int \sqrt{25\cosh^2 u - 25} \times \frac{5}{2}\sinh u\, du$$

$$= \frac{25}{2}\int \sinh^2 u\, du$$

$$= \frac{25}{2}\int \left(\frac{1}{2}\cosh 2u - \frac{1}{2}\right)\, du$$

$$= \frac{25}{4}\left[\frac{1}{2}\sinh 2u - u\right] + c$$

$$= \frac{25}{4}\, [\sinh u \cosh u - u] + c$$

$$= \frac{25}{4}\left[\frac{2}{5}x\sqrt{\frac{4}{25}x^2 - 1} - \operatorname{arcosh}\left(\frac{2}{5}x\right)\right] + c$$

$$= \frac{1}{2}x\sqrt{4x^2 - 25} - \frac{25}{4}\operatorname{arcosh}\left(\frac{2}{5}x\right) + c$$

Question 7

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

$$let\ \frac{1+x}{1-x} = 3 \Rightarrow 1 + x = 3 - 3x \Rightarrow x = \frac{1}{2}$$

$$When\ x = \frac{1}{2} \Rightarrow \ln(3) = 2\left(\frac{1}{2}\right) + \frac{2\left(\frac{1}{2}\right)^3}{3} + \frac{2\left(\frac{1}{2}\right)^5}{5} + \cdots$$

$$\ln(3) = 1 + \frac{1}{3\times 4^1} + \frac{1}{5\times 4^2} + \cdots = \sum_0^{\infty} \frac{1}{(2r+1)\, 4^r}$$

Question 8

$$f(x) = \arctan\left(\sqrt{2} + x\right) \qquad \Rightarrow f(0) = \arctan\sqrt{2}$$

$$f'(x) = \frac{1}{1 + \left(\sqrt{2} + x\right)^2} \qquad \Rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{-2\left(\sqrt{2} + x\right)}{\left\{1 + \left(\sqrt{2} + x\right)^2\right\}^2} \quad \Rightarrow f''(0) = -\frac{2\sqrt{2}}{9}$$

$$f(x) = \arctan\sqrt{2} + \frac{x}{3 \times 1!} - \frac{2\sqrt{2}\,x^2}{9 \times 2!} + \cdots$$

$$f(x) = \arctan\sqrt{2} + \frac{x}{3} - \frac{\sqrt{2}\,x^2}{9} + \cdots$$

Now

$$\int_{-h}^{h} x \arctan\left(\sqrt{2} + x\right)\, dx = \int_{-h}^{h} x\left(\arctan\sqrt{2} + \frac{x}{3} - \frac{\sqrt{2}\,x^2}{9} + \cdots\right) dx$$

$$= \int_{-h}^{h} \left(x \arctan\sqrt{2} + \frac{x^2}{3} - \frac{\sqrt{2}\,x^3}{9} + \cdots\right) dx$$

$$= \left[\frac{x^2}{2} \arctan\sqrt{2} + \frac{x^3}{9} - \frac{\sqrt{2}\,x^4}{36} + \cdots\right]_{-h}^{h}$$

$$= \left(\frac{h^2}{2} \arctan\sqrt{2} + \frac{h^3}{9} - \frac{\sqrt{2}\,h^4}{36} + \cdots\right) - \left(\frac{h^2}{2} \arctan\sqrt{2} - \frac{h^3}{9} - \frac{\sqrt{2}\,h^4}{36} + \cdots\right) \approx \frac{2h^3}{9}$$

Question 9

$$sin(x) = x - \frac{x^3}{3!} + \cdots$$

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$Now \tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \frac{x - \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \cdots\right)\left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)\right]^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \cdots\right)\left[1 - (-1)\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right) + \frac{(-1)\times(-2)}{2!}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right]$$

$$= \left(x - \frac{x^3}{6} + \cdots\right)\left[1 + \frac{x^2}{2} \ldots\right]$$

$$= x - \frac{x^3}{6} + \frac{x^3}{2} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \cdots$$