Edexcel A Level FM Revision Questions: Solutions

Sequences, series; induction; roots of equations (real)

Question 1

For the recurrence relation $u_{n+1} = u_n + 2^{n+1}$ ------ (A) with $u_1 = 5$

Prove true for n = 1 $u_1 = 5$ $u_1 = 2^{1+1} + 1 = 5$ check

Assume statement is true for n = k ⇒ $u_k = 2^{k+1} + 1$ ------- (B)

Prove that the statement is true for n = k + 1

We need to prove the statement ⇒ $u_{k+1} = 2^{k+2} + 1$ ----- (C)

Consider $LHS\ u_{k+1}$ $u_{k+1} = u_k + 2^{k+1}$ from (A)

$u_{k+1} = 2^{k+1} + 1 + 2^{k+1}$ using (B)

$u_{k+1} = 2 \times 2^{k+1} + 1$ ⇒ $u_{k+1} = 2^1 \times 2^{k+1} + 1$

$u_{k+1} = 2^{k+2} + 1$ which is statement C as required … thus proved.

Question 2 $1 + 8 + 27 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

Prove statement true for n = 1 LHS = 1 RHS = $\frac{1}{4}1^2(1+1)^2 = 1$ check

Assume statement true for n = k

$$1 + 8 + 27 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2 \qquad ---(A)$$

Prove statement true for n = k + 1

We need to show that

$$1+8+27+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2 \qquad ---(B)$$

$$1+8+27+\cdots+k^3+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3 \qquad from\ statement\ (A)$$

$$=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$

$$=\frac{1}{4}(k+1)^2\{k^2+4(k+1)\}$$

$$1+8+27+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2\ \ which\ is\ statement\ B\ thus\ proved$$

Question 3

Prove statement true for n = 0

$$5^0+2\times 11^0=3 \qquad check$$

Assume statement true for n = k

$$5^k+2\times 11^k=3\times P \quad ---(A)$$

Prove statement true for n = k + 1

$We\ need\ to\ show\ that\ \ 5^{k+1}+2\times 11^{k+1}\ is\ divisible\ by\ 3$

$$5^{k+1}+2\times 11^{k+1}=5\times 5^k+2\times 11^{k+1}$$

$$=5\times\left(3\times P-2\times 11^k\right)+2\times 11^{k+1} \qquad from\ statement\ (A)$$

$$=5\times 3\times P-10\times 11^k+2\times 11^{k+1}$$

$$=5\times 3\times P-10\times 11^k+2\times 11\times 11^k$$

$$=5\times 3\times P+12\times 11^k$$

$$5^{k+1}+2\times 11^{k+1}=3\times\left(5\times P+4\times 11^k\right)\ which\ is\ divisible\ by\ 3\ so\ styatement\ proved.$$

Question 4

(a) $\alpha+\beta+\gamma=\left\{-\frac{b}{a}\right\}\Rightarrow\ 3=-p \qquad \Rightarrow\ p=-3$

$\alpha\beta+\beta\gamma+\alpha\gamma=\left\{\frac{c}{a}\right\} \qquad \Rightarrow \alpha\beta+\beta\gamma+\alpha\gamma=q$

$Now\ (\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma) \qquad \Rightarrow\ 9=4+2q \qquad \Rightarrow\ q=\frac{5}{2}$

(b) roots of $2x^3-5x^2+ux+v=0$ are $\mathrm{w}, 10, -6\mathrm{w}$

$$\alpha+\beta+\gamma=\left\{-\frac{b}{a}\right\} \qquad \Rightarrow \quad w+10w-6w=\frac{5}{2} \qquad \Rightarrow \quad 5w=\frac{5}{2} \qquad \Rightarrow \quad w=\frac{1}{2}$$

Roots are $\frac{1}{2}$, $5\ and-3$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\left\{\frac{c}{a}\right\} \qquad \Rightarrow \quad \frac{1}{2}\times 5+5\times -3+\frac{1}{2}\times 3=\frac{u}{2} \qquad \Rightarrow \quad u=-28$$

$$\alpha\beta\gamma=\left\{\frac{d}{a}\right\} \qquad \Rightarrow \quad \frac{1}{2}\times 5\times -3=\frac{v}{2} \qquad \Rightarrow \quad v=15$$

<u>Question 5</u>

The roots of the cubic equation $x^3-4x^2+8x+7=0$ are α, β and γ.

Find the cubic equation whose roots are $2\alpha+1, 2\beta+1$ and $2\gamma+1$.

The new roots (y) are related to the old roots (x) by the transformation $y\ =\ 2x+\ 1$

Thus to transform x → y, we need to substitute into the equation for the old roots

$$x=\frac{y-1}{2}$$

to give the equation of the new roots.

$$\left(\frac{y-1}{2}\right)^3-4\left(\frac{y-1}{2}\right)^2+8\left(\frac{y-1}{2}\right)+7=0$$

X 8

$$8\left(\frac{y-1}{2}\right)^3-32\left(\frac{y-1}{2}\right)^2+64\left(\frac{y-1}{2}\right)+56=0$$

$$y^3-3y^2+3y-1-8y^2+16y-8+32y-32+56=0$$

$$y^3-11y^2+51y+15=0$$

<u>Question 6</u>

$$\sum_{r=1}^{n} r^2(3-4r)=\frac{1}{2}n(n+1)(1-2n^2)$$

$$\sum_{r=1}^{n} r^2(3-4r)=3\sum_{r=1}^{n} r^2-4\sum_{r=1}^{n} r^3$$

$$\sum_{r=1}^{n} r^2(3-4r)=3\times\frac{1}{6}n(n+1)(2n+1)-4\times\frac{1}{4}n^2(n+1)^2$$

$$\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)(2n+1) - n^2(n+1)^2$$

$$\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)[(2n+1) - 2n(n+1)]$$

$$\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)(1+2n-2n-2n^2)$$

$$\sum_{r=1}^{n} r^2(3-4r) = \frac{1}{2}n(n+1)(1-2n^2)$$

Question 7

$$\frac{1}{r(r+2)} = \frac{\left(\frac{1}{2}\right)}{r} - \frac{\left(\frac{1}{2}\right)}{r+2}$$

.

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)}{r} - \sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)}{r+2}$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

$A = 3 \text{ and } B = 5$

Question 8

$$2r + 5 = A(2r+1)(2r+3) + B(2r-1)(2r+3) + C(2r-1)(2r+1)$$

$Let\ r = \frac{1}{2} \quad \Rightarrow 6 = A \times 2 \times 4 \quad \Rightarrow A = \frac{3}{4}$

$Let\ r = -\frac{1}{2} \quad \Rightarrow 4 = B \times -2 \times 2 \quad \Rightarrow B = -1$

$Let\ r = -\frac{3}{2} \quad \Rightarrow 2 = C \times -4 \times -2 \quad \Rightarrow C = \frac{1}{4}$

$Thus\ A + B + C = 0$

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \frac{3}{4\times 1} + \frac{3}{4\times 3} + \frac{3}{4\times 5} + \cdots + \frac{3}{4(2n-3)} + \frac{3}{4(2n-1)}$$

$$= \quad -\frac{1}{3} \quad -\frac{1}{5} \quad - \cdots - \frac{1}{(2n-3)} - \frac{1}{(2n-1)} - \frac{1}{(2n+1)}$$

$$= \quad +\frac{1}{4\times 5} + \cdots + \frac{1}{4(2n-3)} + \frac{1}{4(2n-1)} + \frac{1}{4(2n+1)}$$

$$+\frac{1}{4(2n+3)}$$

$$\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$

$$= \frac{2}{3} \qquad - \frac{3}{4(2n+1)}$$

$$+\frac{1}{4(2n+3)}$$

Be careful about lining up similar terms to aid the calculation.

$$Thus \quad P = \frac{2}{3}, \quad Q = -\frac{3}{4} \quad and \quad R = \frac{1}{4}$$