Advanced Mathematics Support Programme®

Edexcel A Level FM Revision Questions: Solutions

Matrices and Vectors

Solutions

1. Invariant points satisfy $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 5x + 4y = x, -4x - 3y = y \ \Rightarrow y = -x$

This is a line of invariant points.

For points on the line $y = mx + c$,

$\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 5x + 4mx + 4c \\ -4x - 3mx - 3c \end{pmatrix}$

For this to be an invariant line,

$-4x - 3mx - 3c = m(5x + 4mx + 4c) + c$ for all x

Equating coefficients of x, $-4 - 3m = 5m + 4m^2$,

so that $4m^2 + 8m + 4 = 0$, or $m^2 + 2m + 1 = 0$;

ie $(m + 1)^2 = 0$, so that $m = -1$

Equating the constant terms, $-3c = 4mc + c$

$\Rightarrow c(4m + 4) = 0$

$\Rightarrow c = 0$ or $m = -1$

So the overall condition is: $m = -1$ and c can take any value,

and the invariant lines are of the form $y = c - x$ (including the line of invariant points $y = -x$).

[Note: $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$ represents a shear, as its determinant is 1 and the sum of the elements on the leading diagonal is 2. It follows that the invariant lines will all be parallel to the line of invariant points.]

2. First of all, none of the lines are parallel to each other.

Then $\begin{vmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{-2} \\ \mathbf{1} & \mathbf{17} & \mathbf{-25} \end{vmatrix} = \mathbf{1(-91) - (-1)(-98) + 3(63) = 0}$

[as expected for this sort of question]

So the planes will either be configured as a sheaf (if they have a line of intersection) or as a triangular prism (if not).

[In some cases it may be possible to spot that one equation is a combination of the other two, showing that the equations are consistent, and that they meet in a line.]

$\boldsymbol{x - y + 3z = 4}$ (1)
$\boldsymbol{4x + 5y - 2z = 8}$ (2)
$\boldsymbol{x + 17y - 25z = -12}$(3)

Substituting for $\boldsymbol{x}$ (say), from (1) into (2) gives:

$\boldsymbol{4(4 + y - 3z) + 5y - 2z = 8}$, so that $\boldsymbol{9y - 14z = -8}$

Substituting into (3) gives:

$\boldsymbol{(4 + y - 3z) + 17y - 25z = -12}$, so that $\boldsymbol{18y - 28z = -16}$,

which is the same equation, and hence the planes meet as a sheaf.

To find the line of intersection, let $\boldsymbol{x = \lambda}$ (say).

Then, from (1), $\boldsymbol{-y + 3z = 4 - \lambda}$ (3)

and from (2), $\boldsymbol{5y - 2z = 8 - 4\lambda}$ (4)

Then $\boldsymbol{5(3) + (4) \Rightarrow 13z = 28 - 9\lambda}$

and $\boldsymbol{2(3) + 3(4) \Rightarrow 13y = 32 - 14\lambda}$,

so that the equation of the line is $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \boldsymbol{\frac{1}{13}} \begin{pmatrix} \boldsymbol{13\lambda} \\ \boldsymbol{32 - 14\lambda} \\ \boldsymbol{28 - 9\lambda} \end{pmatrix}$

or $\boldsymbol{\frac{x}{13} = \frac{y - \frac{32}{13}}{-14} = \frac{z - \frac{28}{13}}{-9}}$

[As a check, points on the line where $\boldsymbol{\lambda = 0 \text{ and } 1}$ could be substituted into the equations of the planes.

Also, it can be shown that the determinant formed by replacing (any) one of the columns of the matrix by the right-hand values will be zero when the equations are consistent. (Consider the $\boldsymbol{2 \times 2}$ case to see why this is likely to be true.)

Thus $\begin{vmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{4} & \mathbf{5} & \mathbf{8} \\ \mathbf{1} & \mathbf{17} & \mathbf{-12} \end{vmatrix} = \mathbf{1(-196) - (-1)(-56) + 4(63) = 0}$, for example.]

3. (i)(a) The angle between the line and the normal to the plane is given by

$\begin{pmatrix} \mathbf{2} \\ \mathbf{-3} \\ \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{pmatrix} = \sqrt{\mathbf{14}}\sqrt{\mathbf{6}}\,\boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{-3}}{\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}} = \mathbf{-0.32733}$

and $\boldsymbol{\theta} = \mathbf{109.107}°$

The acute angle between these vectors is then $\mathbf{180 - 109.107 = 70.893}°$

The acute angle between the line and plane is then

$\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(b) $\begin{pmatrix} \mathbf{-2} \\ \mathbf{3} \\ \mathbf{-1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{pmatrix} = \sqrt{\mathbf{14}}\sqrt{\mathbf{6}}\,\boldsymbol{cos\theta} \Rightarrow \boldsymbol{cos\theta} = \frac{\mathbf{3}}{\sqrt{\mathbf{14}}\sqrt{\mathbf{6}}} = \mathbf{0.32733}$

and $\boldsymbol{\theta} = \mathbf{70.893}°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(ii) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\,\boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{-15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} = \mathbf{-0.69338}$

and $\boldsymbol{\theta} = \mathbf{133.898}°$

The acute angle between the planes themselves is $\mathbf{180 - 133.898 = 46.1}°$

(ii)(b) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{-4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\,\boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} = \mathbf{0.69338}$

and $\boldsymbol{\theta} = \mathbf{46.1}°$

The acute angle between the planes is also $\mathbf{46.1}°$.

4. Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2 + 3\lambda) - 2(4\lambda) + (-1 + \lambda) = 4 \;\Rightarrow -4\lambda = 3;\; \lambda = -\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda = 1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5 + \lambda) - 2(4 - 2\lambda) + (\lambda) = 4 \;\Rightarrow 6\lambda = 7;\; \lambda = \frac{7}{6}$$

So R is $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and Q' will be $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + 2\left(\frac{7}{6}\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$

Then , as P is $\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix} + \lambda\left[\begin{pmatrix} \frac{22}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}\right] = \frac{1}{12}\begin{pmatrix} -3+\lambda(88+3) \\ -36+\lambda(-8+36) \\ -21+\lambda(28+21) \end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix} -3+91\lambda \\ -36+28\lambda \\ -21+49\lambda \end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91} = \frac{y+\frac{36}{12}}{28} = \frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{1}{4}}{13} = \frac{y+3}{4} = \frac{z+\frac{7}{4}}{7}$

5. **Method 1**

The lines are parallel.

Choose a point on one of the lines; eg $\boldsymbol{P} = (-3, 6, 7)$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ 4 \end{pmatrix}$

Then $\overrightarrow{PQ} = \begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ -8+2\lambda \\ -3 \end{pmatrix}$

We want $\overrightarrow{PQ}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} 2+\lambda \\ -8+2\lambda \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$

$\Rightarrow 2+\lambda-16+4\lambda = 0 \quad \Rightarrow 5\lambda = 14; \lambda = \frac{14}{5}$

Then $\overrightarrow{PQ} = \begin{pmatrix} \frac{24}{5} \\ -\frac{12}{5} \\ -\frac{15}{5} \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 8 \\ -4 \\ -5 \end{pmatrix}$ and the required distance is $\frac{3}{5}\sqrt{64+16+25}$

$= \frac{3\sqrt{105}}{5}$

Method 2

Choose a point on each line; eg $\boldsymbol{R} = (-1, -2, 4)$ on the 1st line, and

$\boldsymbol{P} = (\mathbf{-3, 6, 7})$ on the 2nd line.

Then $\overrightarrow{\boldsymbol{PR}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-8} \\ \mathbf{-3} \end{pmatrix}$ and the required distance is $\left| \dfrac{\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right|} \right|$

$$= \left| \frac{\begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & -8 & 2 \\ \underline{k} & -3 & 0 \end{vmatrix}}{\sqrt{5}} \right| = \frac{1}{\sqrt{5}} \left| \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \left| \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \sqrt{21} = \frac{3\sqrt{105}}{5}$$

6. (i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -3 + 5\lambda \\ 2 + 3\lambda \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 4 + 2\mu \\ -1 + 2\mu \end{pmatrix}$$

A point of intersection would then satisfy

$\mathbf{1 + 2\lambda = \mu}$ (1)
$\mathbf{-3 + 5\lambda = 4 + 2\mu}$ (2)
$\mathbf{2 + 3\lambda = -1 + 2\mu}$ (3)

Substituting from (1) into (2) & (3) gives:

$\mathbf{-3 + 5\lambda = 4 + 2(1 + 2\lambda)}$ or $\mathbf{-9 = -\lambda}$, so that $\boldsymbol{\lambda} = \mathbf{9}$

and $\mathbf{2 + 3\lambda = -1 + 2(1 + 2\lambda)}$ or $\mathbf{1 = \lambda}$,

and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{\boldsymbol{OX}} = \begin{pmatrix} 1 + 2\lambda \\ -3 + 5\lambda \\ 2 + 3\lambda \end{pmatrix} \text{ and } \overrightarrow{\boldsymbol{OY}} = \begin{pmatrix} \mu \\ 4 + 2\mu \\ -1 + 2\mu \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{\boldsymbol{XY}}$ will be perpendicular to both lines, so that

$\overrightarrow{XY}.\begin{pmatrix}2\\5\\3\end{pmatrix} = 0$ and $\overrightarrow{XY}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0$, so that

$$\begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}2\\5\\3\end{pmatrix} = 0 \text{ and } \begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0,$$

giving $(2\mu-2-4\lambda)+(35+10\mu-25\lambda)+(-9+6\mu-9\lambda) = 0$
or $18\mu-38\lambda = -24$; ie $9\mu-19\lambda = -12$ (1)

and $(\mu-1-2\lambda)+(14+4\mu-10\lambda)+(-6+4\mu-6\lambda) = 0$

$9\mu-18\lambda = -7$ (2)

Then $(1)-(2) \Rightarrow -\lambda = -5$, so that $\lambda = 5$ and, from (2),

$\mu = \frac{1}{9}(18(5)-7) = \frac{83}{9}$

So $\overrightarrow{OX} = \begin{pmatrix}11\\22\\17\end{pmatrix}$ and $\overrightarrow{OY} = \frac{1}{9}\begin{pmatrix}83\\202\\157\end{pmatrix}$

and $\overrightarrow{XY} = \frac{1}{9}\begin{pmatrix}83-99\\202-198\\157-153\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-16\\4\\4\end{pmatrix} = \frac{4}{9}\begin{pmatrix}-4\\1\\1\end{pmatrix}$,

so that $|\overrightarrow{XY}| = \frac{4}{9}\sqrt{16+1+1} = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

Method 2 (using the vector product)

If $\underline{\hat{n}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{OX}$ and $\overrightarrow{OY}$ such that $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$, and the shortest distance will then be $|d|$.

A vector perpendicular to both lines is $\begin{pmatrix}2\\5\\3\end{pmatrix} \times \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{vmatrix}\underline{i} & 2 & 1\\ \underline{j} & 5 & 2\\ \underline{k} & 3 & 2\end{vmatrix}$

$= \begin{pmatrix}4\\-1\\-1\end{pmatrix}$, so that $\underline{\hat{n}} = \frac{1}{\sqrt{18}}\begin{pmatrix}4\\-1\\-1\end{pmatrix}$

Then $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$ gives $\begin{pmatrix}1+2\lambda\\-3+5\lambda\\2+3\lambda\end{pmatrix} + D\begin{pmatrix}4\\-1\\-1\end{pmatrix} = \begin{pmatrix}\mu\\4+2\mu\\-1+2\mu\end{pmatrix}$,

where $D = \frac{d}{\sqrt{18}}$,

so that $\mathbf{2}\boldsymbol{\lambda} + \mathbf{4}\boldsymbol{D} - \boldsymbol{\mu} = -\mathbf{1}$ (1)
$\mathbf{5}\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = \mathbf{7}$ (2)
$3\boldsymbol{\lambda} - \boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = -\mathbf{3}$ (3)

Then $(\mathbf{2}) - (\mathbf{3}) \Rightarrow \mathbf{2}\boldsymbol{\lambda} = \mathbf{10}$, so that $\boldsymbol{\lambda} = \mathbf{5}$

and (1) & (2) become $\mathbf{4}\boldsymbol{D} - \boldsymbol{\mu} = -\mathbf{11}$ (4) and $-\boldsymbol{D} - \mathbf{2}\boldsymbol{\mu} = -\mathbf{18}$ (5)

Then $\mathbf{2}(\mathbf{4}) - (\mathbf{5}) \Rightarrow \mathbf{9}\boldsymbol{D} = -\mathbf{4}$, so that $|\boldsymbol{d}| = \sqrt{\mathbf{18}}|\boldsymbol{D}| = \frac{\mathbf{4}\sqrt{\mathbf{18}}}{\mathbf{9}} = \frac{\mathbf{4}\sqrt{\mathbf{2}}}{\mathbf{3}}$

and, from (1), $\boldsymbol{\mu} = \mathbf{10} - \frac{\mathbf{16}}{\mathbf{9}} + \mathbf{1} = \frac{\mathbf{83}}{\mathbf{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).