Advanced Mathematics Support Programme®

Edexcel A Level FM Revision Questions: Solutions

Integration

Solutions

1.(a) $\frac{2x^2-x+5}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

so that $2x^2 - x + 5 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$

$x = -2 \Rightarrow 15 = -5C; C = -3$

$x = 3 \Rightarrow 20 = 25A; A = \frac{4}{5}$

Equating coefficients of $x^0: 5 = 4A - 6B - 3C$

$\Rightarrow B = \frac{1}{6}\left(\frac{16}{5} + 9 - 5\right) = \frac{6}{5}$

[Check: Equating coefficients of $x^2: 2 = A + B$

and $x = 1 \Rightarrow 6 = 9A - 6B - 2C$]

Then $\int \frac{2x^2-x+5}{(x-3)(x+2)^2}\, dx = \int \frac{4}{5(x-3)} + \frac{6}{5(x+2)} - \frac{3}{(x+2)^2}\, dx$

$= \frac{4}{5} ln|x-3| + \frac{6}{5} ln|x+2| + \frac{3}{x+2} + c$

(b) As the degree of the numerator is not less than that of the denominator, a rearrangement is necessary, in order for the standard approach to be applied. This is done in Method 1, although Method 2 is quicker.

Method 1

$\frac{(x+1)(x^2+1)}{(x+2)(x^2+2)} = \frac{x^3+x^2+x+1}{x^3+2x^2+2x+4} = \frac{x^3+2x^2+2x+4}{x^3+2x^2+2x+4} - \frac{x^2+x+3}{x^3+2x^2+2x+4}$

$= 1 - \frac{x^2+x+3}{(x+2)(x^2+2)}$

$= 1 - \frac{A}{x+2} - \frac{Bx+C}{x^2+2}$

where $A(x^2+2) + (Bx+C)(x+2) = x^2 + x + 3$

Then setting $x = -2$ gives $6A = 5$, so that $A = \frac{5}{6}$

Also, equating constant terms: $2A + 2C = 3$, so that $C = \frac{1}{2}\left(3 - \frac{5}{3}\right) = \frac{2}{3}$

and equating coefficients of $\boldsymbol{x^2}$: $\boldsymbol{A+B=1}$, so that $\boldsymbol{B=\frac{1}{6}}$

[Check: equating coefficients of $\boldsymbol{x}$: $\boldsymbol{2B+C=1}$

and setting $\boldsymbol{x=1}$ gives $\boldsymbol{3A+3B+3C=5}$]

Method 2

$$\frac{(x+1)(x^2+1)}{(x+2)(x^2+2)}=\frac{x^3+x^2+x+1}{(x+2)(x^2+2)}=1+\frac{D}{x+2}+\frac{Ex+F}{x^2+2}$$

where $\boldsymbol{(x+2)(x^2+2)+D(x^2+2)+(Ex+F)(x+2)=x^3+x^2+x+1}$

Then, setting $\boldsymbol{x=-2}$ gives $\boldsymbol{6D=-5}$, so that $\boldsymbol{D=-\frac{5}{6}}$

Also, equating constant terms: $\boldsymbol{4+2D+2F=1}$, so that $\boldsymbol{F=\frac{1}{2}\left(-3+\frac{5}{3}\right)=-\frac{2}{3}}$

and equating coefficients of $\boldsymbol{x^2}$: $\boldsymbol{2+D+E=1}$, so that $\boldsymbol{E=-\frac{1}{6}}$

Then $\int_0^{\sqrt{2}}\frac{(x+1)(x^2+1)}{(x+2)(x^2+2)}dx=\int_0^{\sqrt{2}}1-\frac{5}{6(x+2)}-\frac{x+4}{6(x^2+2)}dx$

$$=\left[x-\frac{5}{6}\ln(x+2)-\frac{1}{12}\ln(x^2+2)-\frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)\arctan\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$=\left(\sqrt{2}-\frac{5}{6}\ln(2+\sqrt{2})-\frac{1}{12}\ln 4-\frac{\sqrt{2}}{3}\left(\frac{\pi}{4}\right)\right)-\left(-\frac{5}{6}\ln 2-\frac{1}{12}\ln 2\right)$$

$$=\sqrt{2}-\frac{5}{6}\ln(2+\sqrt{2})+\frac{3}{4}\ln 2-\frac{\pi\sqrt{2}}{12}$$

2. (a) As the numerator integrates to e^x and the rest of the integrand is a function of e^x that we can integrate, let $u=e^x$, so that $du=e^x\,dx$.

Then $\int\frac{e^x}{e^{2x}+1}\,dx=\int\frac{1}{u^2+1}\,du=\arctan(e^x)+c$

(b) As the numerator integrates to $\frac{1}{2}x^2$, and $\frac{1}{1+u^2}$ can be integrated,

let $u=x^2$, so that $du=2x\,dx$,

and $\int\frac{x}{1+x^4}\,dx=\frac{1}{2}\int\frac{1}{1+u^2}\,du=\frac{1}{2}\arctan(x^2)+c$

(c) $\int\frac{1}{x^2+6x+18}\,dx=\int\frac{1}{(x+3)^2+9}\,dx=\frac{1}{3}\arctan\left(\frac{x+3}{3}\right)+c$

(d) Let $2x^2 = 3tan^2\theta$, so that $x = \sqrt{\frac{3}{2}}\, tan\theta$ and $dx = \sqrt{\frac{3}{2}}\, sec^2\theta\, d\theta$

Then $I = \int \frac{1}{(2x^2+3)^{\frac{3}{2}}}\, dx = \sqrt{\frac{3}{2}} \int \frac{sec^2\theta}{3^{\frac{3}{2}}(tan^2\theta+1)^{\frac{3}{2}}}\, d\theta$

$= \frac{1}{3\sqrt{2}} \int \frac{sec^2\theta}{sec^3\theta}\, d\theta = \frac{1}{3\sqrt{2}} \int cos\theta\, d\theta = \frac{1}{3\sqrt{2}} sin\theta + c$

Now $sin^2\theta = 1 - cos^2\theta = 1 - \frac{1}{1+tan^2\theta} = 1 - \frac{1}{1+\frac{2}{3}x^2}$

$= 1 - \frac{3}{3+2x^2} = \frac{2x^2}{3+2x^2}$

so that $I = \frac{1}{3\sqrt{2}} \sqrt{\frac{2x^2}{3+2x^2}} + c = \frac{x}{3\sqrt{3+2x^2}} + c$

(e) $\int \frac{4x+5}{\sqrt{4-6x-x^2}}\, dx = -2 \int \frac{-6-2x}{\sqrt{4-6x-x^2}}\, dx - \int \frac{7}{\sqrt{4-6x-x^2}}\, dx$

$= -\frac{2(4-6x-x^2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - 7 \int \frac{1}{\sqrt{13-(x+3)^2}}\, dx$

$= -4(4 - 6x - x^2)^{\frac{1}{2}} - 7arcsin\left(\frac{x+3}{\sqrt{13}}\right) + c$

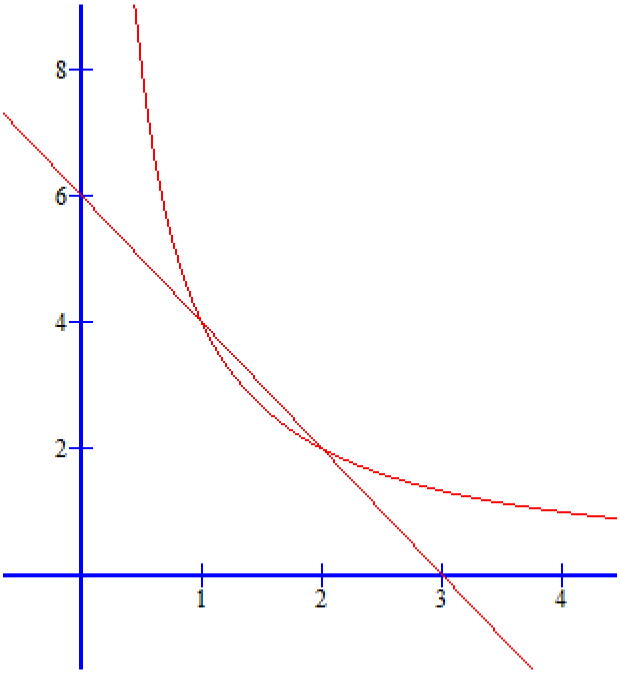
3. To find the points of intersection of the line and curve:

$\boldsymbol{y = 6 - 2x}$ and $\boldsymbol{y = \frac{4}{x} \Rightarrow \frac{4}{x} = 6 - 2x}$,

so that $\boldsymbol{4 = 6x - 2x^2}$, or $\boldsymbol{x^2 - 3x + 2 = 0}$

$\Rightarrow \boldsymbol{(x-1)(x-2) = 0}$,

so that the points of intersection are $(\boldsymbol{1,4})$ and $(\boldsymbol{2,2})$.



For the line, $x = 3 - \frac{y}{2}$, and for the curve, $x = \frac{4}{y}$

The required volume is $\int_2^4 \pi\left\{\left(3 - \frac{y}{2}\right)^2 - \left(\frac{4}{y}\right)^2\right\} dy$

$= \pi \int_2^4 9 - 3y + \frac{y^2}{4} - \frac{16}{y^2}\ dy$

$= \pi[9y - \frac{3y^2}{2} + \frac{y^3}{12} + \frac{16}{y}]_2^4$

$= \pi\{\left(36 - 24 + \frac{16}{3} + 4\right) - \left(18 - 6 + \frac{2}{3} + 8\right)\}$

$= \pi\left\{-4 + \frac{14}{3}\right\} = \frac{2\pi}{3}\ units^3$

4. (i)(a) Volume $= \pi \int_0^1 y^2 dx = \pi \int_0^1 4x\, dx = \pi[2x^2]_0^1 = 2\pi(1 - 0) = 2\pi$

(b) $x = 0 \Rightarrow t = 0; x = 1 \Rightarrow t = 1$

Volume $= \pi \int_0^1 y^2 \frac{dx}{dt} dt = \pi \int_0^1 (2t)^2(2t)dt = 8\pi \int_0^1 t^3 dt$

$= 8\pi \left[\frac{1}{4}t^4\right]_0^1 = 2\pi(1 - 0) = 2\pi$

(ii) Mean value $= \frac{1}{1-0}\int_0^1 \sqrt{4x}\, dx = 2\int_0^1 x^{\frac{1}{2}}\, dx$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{4}{3}$$

Approximate volume is that of a cylinder of radius $\frac{4}{3}$ and length 1;

ie $\pi\left(\frac{4}{3}\right)^2(1) = \frac{16}{9}\pi$, which is reasonably close to 2π.