Advanced Mathematics Support Programme®

Edexcel A Level FM Revision Questions: Solutions

Differential equations; SHM

Solution 1

$$\frac{dy}{dx} - \frac{x}{x+1}y = \frac{e^{2x}}{x+1}$$

$$I.F. = e^{\int -\frac{x}{x+1}dx} = e^{\int -1+\frac{1}{x+1}dx} = e^{-x+\ln(x+1)}$$

$$= e^{\ln(e^{-x})+\ln(x+1)} = e^{\ln(e^{-x})+\ln(x+1)} = e^{\ln[e^{-x}(x+1)]} = e^{-x}(x+1)$$

$$e^{-x}(x+1)\frac{dy}{dx} - e^{-x}x\,y = e^{x}$$

$$\frac{d}{dx}[e^{-x}(x+1)y] = e^{x}$$

$$e^{-x}(x+1)y = \int e^{x}\,dx$$

$$e^{-x}(x+1)y = e^{x} + C$$

$$y = \frac{e^{x}(e^{x}+C)}{(x+1)}$$

$$when\ x = 0\ then\ y = -3 \Rightarrow -3 = 1 + C \Rightarrow C = -4$$

$$y = \frac{e^{x}(e^{x}-4)}{(x+1)}$$

Question 2

(i)

$$\frac{dv}{dt} + \frac{2}{(t+1)}v = g - \frac{2}{(t+1)}$$

$$IF = e^{\int\frac{2}{(t+1)}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

$$(t+1)^2\frac{dv}{dt} + 2(t+1)v = g(t+1)^2 - 2(t+1)$$

$$\frac{d}{dt}[(t+1)^2\,v] = g(t+1)^2 - 2(t+1)$$

$$(t+1)^2\, v = \int g(t+1)^2 - 2(t+1)\, dt$$

$$(t+1)^2\, v = \frac{1}{3} g(t+1)^3 - (t+1)^2 + C$$

$$v\ = \frac{1}{3}(t+1)g - 1 + C(t+1)^{-2}$$

$$When\ t = 0\ then\ v = 0 \Rightarrow C = 1 - \frac{1}{3} g$$

$$v\ = \frac{1}{3}(t+1)g - 1 + \left(1 - \frac{1}{3} g\right)(t+1)^{-2}$$

2 (ii)

$$\frac{dv}{dt} + \frac{4}{(t+1)} v = g$$

$$IF = e^{\int \frac{4}{(t+1)} dt} = e^{4\ln(t+1)} = e^{\ln(t+1)^4} = (t+1)^4$$

$$(t+1)^4 \frac{dv}{dt} + 4(t+1)^3 v = g(t+1)^4$$

$$\frac{d}{dt}[(t+1)^4\, v] = g(t+1)^4)$$

$$\boldsymbol{(t+1)^4\, v = \int g(t+1)^4\, dt}$$

$$(t+1)^4\, v = \frac{1}{5} g(t+1)^5 + C$$

$$v\ = \frac{1}{5}(t+1)g + C(t+1)^{-4}$$

$$When\ t = 0\ then\ v = 0 \Rightarrow C = -\frac{1}{5} g$$

$$v\ = \frac{1}{5}(t+1)g - \frac{1}{5} g(t+1)^{-4}$$

Question 3

(i) $\boldsymbol{Auxillary\ equation\ is\ m^2 + 25 = 0}$

$\boldsymbol{\Rightarrow m = \pm 5i}$

$\boldsymbol{Complementary\ solution\ is\ y = A\cos 5t + B\sin 5t}$

$\boldsymbol{The\ sytem\ will\ exhibit\ simple\ harmonic\ motion\ SHM\ with\ frequency\ \frac{2\,\pi}{5}}$

(ii) $\boldsymbol{Let\ Particular\ Integral\ PI\ be\quad y = P\cos 4t + Q\sin 4t}$

$$\boldsymbol{\Rightarrow \frac{dy}{dt} = -4P\sin 4t + 4Q\cos 4t}$$

$$\Rightarrow \frac{d^2y}{dt^2} = -16P\cos 4t - 16\,Q\sin 4t$$

$$subst.\,into\,\frac{d^2y}{dt^2} + 25y = 41\cos 4t \;\Rightarrow = -16P\cos 4t - 16\,Q\sin 4t + 25\,P\cos 4t + 25\,Q\sin 4t$$
$$= 41\,cos\,4t$$

$$\Rightarrow\; -16P + 25P = 41 \Rightarrow P = \frac{41}{9}$$

$$\Rightarrow\; -16Q + 25Q = 0 \Rightarrow Q = \;0$$

$$Thus\;PI\;is\;y = \frac{41}{9}\cos 4t$$

$$and\;the\;general\;solution\;is\;y = A\cos 5t + B\sin 5t + \frac{41}{9}\cos 4t$$

(iii) $when\;y = 1\;and\;t = 0 \Rightarrow \;\; 1 = A + \frac{41}{9} \Rightarrow A = -\,\frac{32}{9}$

$$\frac{dy}{dt} = -5A\sin 5t + 5B\cos 5t - \frac{41\times 4}{9}\sin 4t$$

$$when\;t = 0\;then\;\frac{dy}{dt} = 0 \Rightarrow B = 0$$

$$Particular\;solution\;is\;\;y = \frac{41}{9}\cos 4t - \frac{32}{9}\cos 5t$$

(iv)

$The\;system\;exhibits\;oscillations\;of a\;combination\;of\;two\;harmonic\;motions\;with$

$frequencies\;\frac{2\,\pi}{5}\;and\;\frac{\pi}{2}.$

(v) $The\;Auxillary\;equation\;is\;m^2 + 8m + 25 = 0$

$$\Rightarrow m = \;-4 \pm 3i$$

$$Complementary\;solution\;is\;y = e^{-4t}(A\cos 3t + B\sin 3t)$$

$$Let\;Particular\;Integral\;PI\;be\;\;\;y = P\cos 4t + Q\sin 4t$$

$$\Rightarrow \frac{dy}{dt} = -4P\sin 4t + 4Q\cos 4t$$

$$\Rightarrow \frac{d^2y}{dt^2} = -16P\cos 4t - 16\,Q\sin 4t$$

$$subst.\,into\,\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 221\cos 4t$$

$$\Rightarrow = -16\,P\cos 4t - 16\,Q\sin 4t - 32\,\mathrm{P}\sin 4t + 32\,\mathrm{Q}\cos 4t + 25\,P\cos 4t + 25\,Q\sin 4t$$
$$= 41\,cos\,4t$$

$\Rightarrow\ -16Q - 32P + 25Q = 0 \quad \Rightarrow P = \dfrac{9}{32}Q$

$\Rightarrow\ -16P + 32Q + 25P = 41 \Rightarrow 9P + 32Q = 41$

$\Rightarrow P = \dfrac{369}{1105}$ *and* $Q = \dfrac{41}{1105}$

Thus PI is $y = \dfrac{369}{1105}\cos 4t + \dfrac{1312}{1105}\sin 4t$

and the general solution is $y = e^{-4t}(A\cos 3t + B\sin 3t) + \dfrac{369}{1105}\cos 4t + \dfrac{1312}{1105}\sin 4t$

(vi) ***The system exhibits oscillations of combination of two harmonic motions with***

frequencies $\dfrac{\pi}{2}$ ***and*** $\dfrac{2\pi}{3}$

but the 2nd harmonic motion is damped.

(The damping ratio is 0.8 so the damping is subcritical.)

Question 4

Auxillary equation is $m^2 + 6m + 9 = 0 \Rightarrow m = -3$ ***and*** -3

Complementary function is $y = (A + Bx)e^{-3x}$

Let PI be $y = \lambda e^{2x}$

$\Rightarrow \dfrac{dy}{dx} = 2\lambda e^{2x}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 4\lambda e^{2x}$

subst. into $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9 = 5e^{2x}$ *gives* $4\lambda + 12\lambda + 9\lambda = 5 \Rightarrow \lambda = \dfrac{1}{5}$

so general solution is $y = (A + Bx)e^{-3x} + \dfrac{1}{5}e^{2x}$

(ii) *when* $x = 0$ *then* $y = 1 \Rightarrow 1 = A + \frac{1}{5} \Rightarrow A = \frac{4}{5}$

$\dfrac{dy}{dx} = -3(A + Bx)e^{-3x} + Be^{-3x} + \dfrac{2}{5}e^{2x}$

when $x = 0$ *then* $\dfrac{dy}{dx} = 0 \Rightarrow 0 = -3A + B + \dfrac{2}{5} \Rightarrow B = 2$

so particular solution is $y = \left(\dfrac{4}{5} + 2x\right)e^{-3x} + \dfrac{1}{5}e^{2x}$

Question 5

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - (5x - 4y + 11)$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - \left\{5x - 4\left(2x - \frac{dx}{dt} + 5\right) + 11\right\}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 9$$

$Auxillary\ equation\ is\ m^2 + 2m - 3 = 0 \Rightarrow m = -3\ and\ m = 1$

$Complementary\ solution\ is\ x = Ae^{-3t} + Be^{t}$

$Let\ PI\ be\ x = \lambda \Rightarrow\ -3\lambda = 9 \Rightarrow\ \lambda =\ -3$

$General\ solution\ is\ x =\ Ae^{-3t} + Be^{t} - 3$

$Now\ y = 2x - \frac{dx}{dt} + 5$

$\Rightarrow\ y = 2(Ae^{-3t} + Be^{t} - 3) - (-3Ae^{-3t} + Be^{t}) + 5$

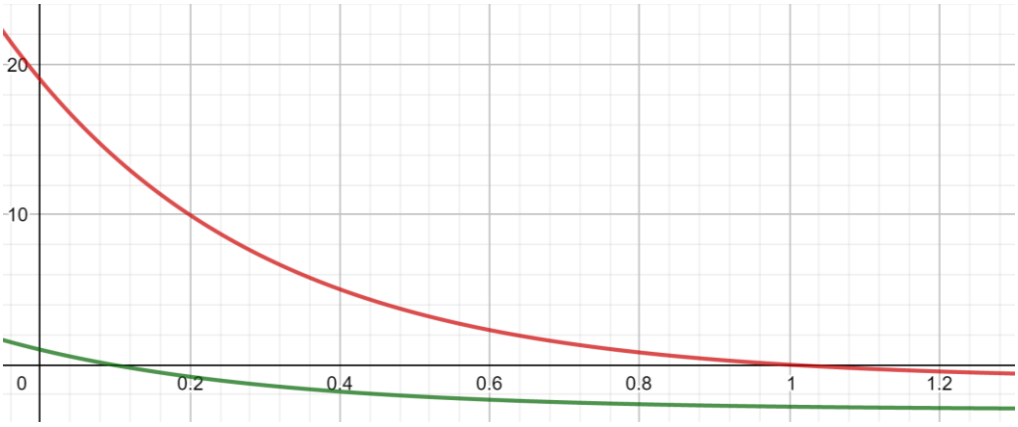
$\Rightarrow\ y = 5Ae^{-3t} + Be^{t} - 1$

$x\ = 1, y\ =\ 19$ when $t\ =\ 0$

$\Rightarrow 4 = A + B$

$\Rightarrow 20 = 5A + B$

$\Rightarrow A = 4\ and\ B = 0 \Rightarrow\ x =\ 4e^{-3t} - 3\ \ and\ \ y = 20e^{-3t} - 1$



The particle moves towards ($\mathbf{-3}$, $\mathbf{-1}$) as $\boldsymbol{t} \to\ \infty$