Edexcel A Level FM Revision Questions: Solutions

Complex numbers; complex roots of equations

Solutions

Question 1

let $p(x) = x^4 - 10x^3 + 54x^2 - 130x + 125 = 0$

If $\mathbf{3 + 4i}$ is a root of p(x) then so is $\mathbf{3 - 4i}$ $\quad\Rightarrow\quad$ $\boldsymbol{x^2 - 6x + 25}$ **is a factor of** $\boldsymbol{p(x)}$

Method 1

As $\mathbf{3 + 4i}$ and $\mathbf{3 - 4i}$ are roots then $\boldsymbol{(x - 3 - 4i)(x - 3 + 4i)}$ is a factor

$\boldsymbol{(x - 3 - 4i)(x - 3 + 4i) = x^2 - 6x + 25}$

Method 2

As $\mathbf{3 + 4i}$ and $\mathbf{3 - 4i}$ are roots then sum of roots = 6 and product of roots = 25 ⇒ $\boldsymbol{x^2 - 6x + 25}$ **is a factor**.

Now

$$\boldsymbol{x^4 - 10x^3 + 54x^2 - 130x + 125 = (x^2 - 6x + 25)(Ax^2 + Bx + C)}$$

Equating x^4 terms: A = 1

Equating x^0 terms: 125 = 25C $\rightarrow$ C = 5

Equating x^1 terms: -130 = -6C + 25B $\rightarrow$ -130 = -30 + 25B $\rightarrow$ B = -4

$$\boldsymbol{x^4 - 10x^3 + 54x^2 - 130x + 125 = (x^2 - 6x + 25)(x^2 - 4x + 5)}$$

Using $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ gives other roots as $\boldsymbol{2 \pm i}$.

Question 2

$\boldsymbol{\sin 5\theta = \mathrm{Im}(\cos 5\theta + i \sin 5\theta)}$

$\qquad \boldsymbol{= \mathrm{Im}\{(\cos\theta + i\sin\theta)^5\}}$

$$= \mathbf{Im}\{\cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\ \sin^4\theta + i\sin^5\theta\}$$

$$= 5\cos^4\theta\ \sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5\,(1-\sin^2\theta)^2\sin\theta - 10\,(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$\sin 5\theta = 5\ \sin\theta\ - 20\ \sin^3\theta\ + 16\sin^5\theta$$

A = 5, B = -20 and C = 16

<u>Question 3</u>

$$z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta$$

Consider

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^4 = \left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\left(z^4-4z^2+6-\frac{4}{z^2}+\frac{1}{z^4}\right)$$

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^4 = \left(z^7+\frac{1}{z^7}\right) - \left(z^5+\frac{1}{z^5}\right) - 3\left(z^3+\frac{1}{z^3}\right) + 3\left(z+\frac{1}{z}\right)$$

$$(2\cos\theta)^3(2i\sin\theta)^4 = (2\cos 7\theta) - (2\cos 5\theta) - 3(\cos 3\theta) + 3(2\cos\theta)$$

$$128\cos^3\theta\sin^4\theta = (2\cos 7\theta) - (2\cos 5\theta) - 3(\cos 3\theta) + 3(2\cos\theta)$$

$$\cos^3\theta\sin^4\theta = \frac{1}{64}\{\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\}$$

Question 4

(a)

$$\left(1+\frac{1}{2}e^{3i\theta}\right)\left(1+\frac{1}{2}e^{-3i\theta}\right) = 1+\frac{1}{2}e^{3i\theta} + \frac{1}{2}e^{-3i\theta} + \frac{1}{4}$$

$$= \frac{5}{4}+\frac{1}{2}(\cos 3\theta + i\sin 3\theta)+\frac{1}{2}(\cos 3\theta - i\sin 3\theta)$$

$$= \frac{5}{4}+\cos 3\theta$$

(b)

$$C+i\,S = \cos 2\theta - \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 8\theta - \frac{1}{8}\cos 11\theta + \cdots$$
$$+\, i\sin 2\theta - \frac{1}{2}i\sin 5\theta + \frac{1}{4}i\sin 8\theta - \frac{1}{8}i\sin 11\theta + \cdots$$

$$\Rightarrow \quad C+i\,S = e^{2i\theta} - \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{8i\theta} - \frac{1}{8}e^{11i\theta} + \cdots$$

Which is a GP with first term $e^{2i\theta}$ and common ratio $-\frac{1}{2}e^{3i\theta}$

$$\Rightarrow \quad C+i\,S = \frac{e^{2i\theta}}{1+\frac{1}{2}e^{3i\theta}}$$

$$\Rightarrow \quad C+i\,S = \frac{e^{2i\theta}\left(1+\frac{1}{2}e^{-3i\theta}\right)}{\left(1+\frac{1}{2}e^{3i\theta}\right)\left(1+\frac{1}{2}e^{-3i\theta}\right)}$$

$$\Rightarrow \quad C+i\,S = \frac{e^{2i\theta}\left(1+\frac{1}{2}e^{-3i\theta}\right)}{\left(\frac{5}{4}+\cos 3\theta\right)}$$

$$\Rightarrow \quad C+i\,S = \frac{4e^{2i\theta} + 2e^{-i\theta}}{5+4\cos 3\theta}$$

$$\Rightarrow \quad C+i\,S = \frac{4(\cos 3\theta + i\sin 3\theta) + 2(\cos\theta - i\sin\theta)}{5+4\cos 3\theta}$$

$$\Rightarrow \quad C = \frac{4\cos 2\theta + 2\cos\theta}{5+4\cos 3\,\theta} \text{ and } S = \frac{4\sin 2\theta - 2\sin\theta}{5+4\cos 3\,\theta}$$

Question 5

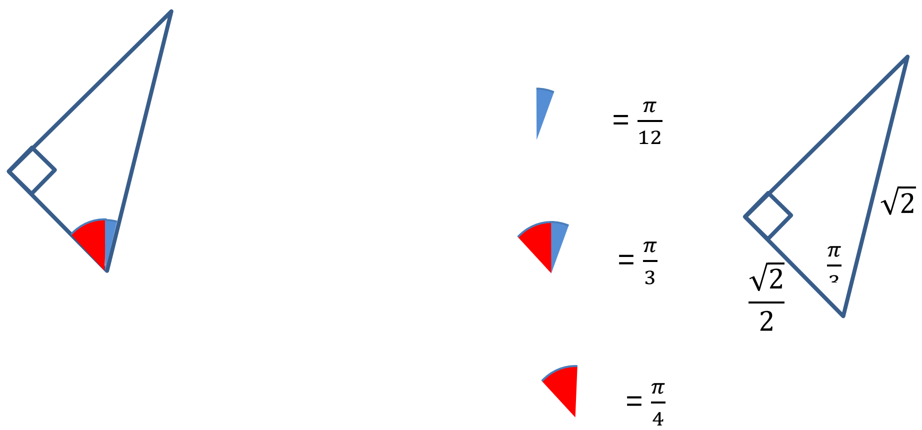
(a) Let $z = -2 - 2i$. Modulus = $2\sqrt{2}$. Argument = $-\frac{3\pi}{4}$

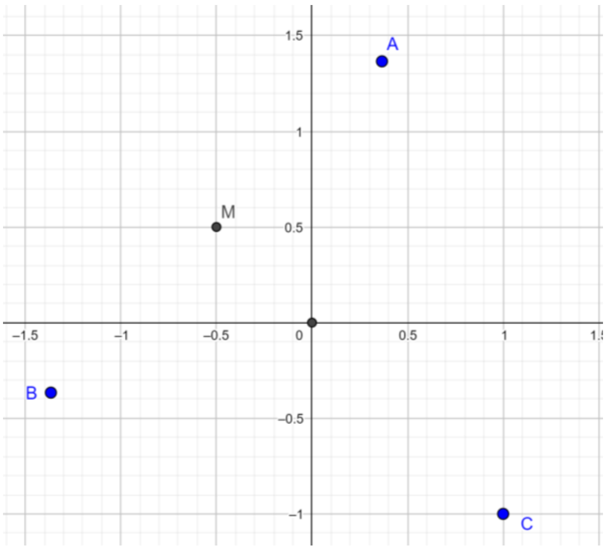
$\Rightarrow z = 2\sqrt{2}e^{-\frac{3\pi}{4}i}$

Cube roots of z are:

$\sqrt{2}\, e^{-\frac{\pi}{4}i}$, $\quad \sqrt{2}\, e^{-\frac{11\pi}{12}i}$, $\quad \sqrt{2}\, e^{\frac{5\pi}{12}i}$

(b)





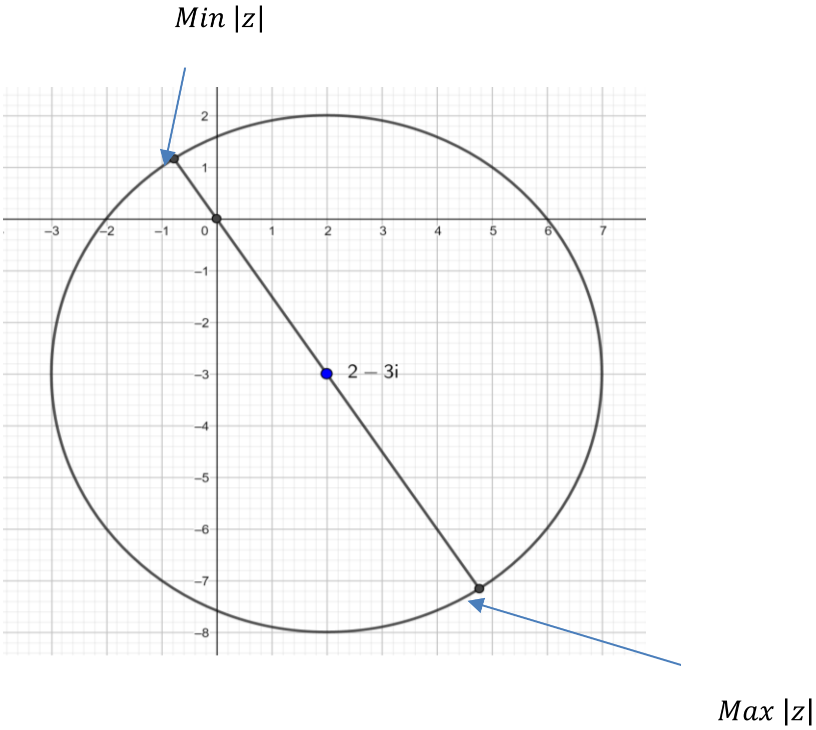
(c) Modulus of $w = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Argument of $w = \frac{3\pi}{4}$

(d) $w^6 = \left(\frac{1}{\sqrt{2}}\right)^6 e^{6\times\frac{3\pi}{4}i} \Rightarrow w^6 = \frac{1}{8}\, e^{\frac{9\pi}{2}i} \Rightarrow w^6 = \frac{1}{8}i$

Question 6

(a)



(b) Min value of $|\boldsymbol{z}| = \mathbf{5} - \sqrt{\mathbf{13}}$

Max value of $|\boldsymbol{z}| = \mathbf{5} + \sqrt{\mathbf{13}}$

Question 7

$|z - 1| = 2|z + 2|$

Let $z = x + iy$

$\Rightarrow \quad |x + iy - 1| = 2|x + iy + 2|$

$\Rightarrow \quad |(x - 1) + iy| = 2|(x + 2) + iy|$

$\Rightarrow \quad (x - 1)^2 + y^2 = 4(x + 2)^2 + 4y^2$

$\Rightarrow \quad x^2 - 2x + 1 + y^2 = 4x^2 + 16x + 16 + 4y^2$

$\Rightarrow \quad 0 = 3x^2 + 18x + 15 + 3y^2$

$\Rightarrow \quad 0 = x^2 + 6x + 5 + y^2$

$\Rightarrow (x+3)^2 + y^2 = 4$

<u>Question 8</u>

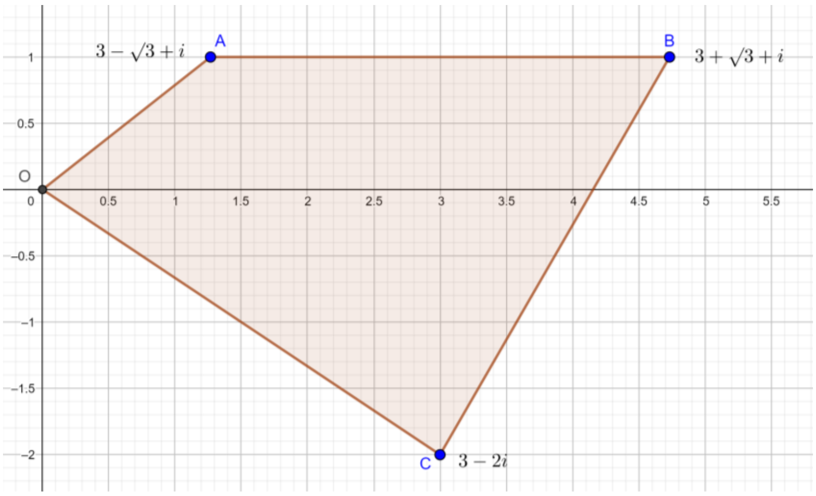
(a)

$$\mathbf{8}\boldsymbol{i} = \mathbf{8}\boldsymbol{e}^{\pi i/2} \Rightarrow \quad \boldsymbol{z} - \mathbf{3} = \quad \mathbf{2}\boldsymbol{e}^{\frac{\pi}{6}i}, \quad \mathbf{2}\boldsymbol{e}^{\frac{5\pi}{6}i}, \quad \mathbf{2}\boldsymbol{e}^{-\frac{\pi}{2}i}$$

$$\Rightarrow \quad \boldsymbol{z} - \mathbf{3} = \quad \sqrt{\mathbf{3}} + \boldsymbol{i}, \quad -\sqrt{\mathbf{3}} + \boldsymbol{i}, \quad -\mathbf{2}\boldsymbol{i}$$

$$\Rightarrow \quad \boldsymbol{z} = \quad \mathbf{3} + \sqrt{\mathbf{3}} + \boldsymbol{i}, \quad \mathbf{3} - \sqrt{\mathbf{3}} + \boldsymbol{i}, \quad \mathbf{3} - \mathbf{2}\boldsymbol{i}$$

(b)



(c) METHOD - Find area of surrounding rectangle and subtract 3 triangles

(other approaches are available)

$$\boldsymbol{Area\ of\ OABC} = (\mathbf{3} + \sqrt{\mathbf{3}}) \times \mathbf{3} - \frac{\mathbf{1}}{\mathbf{2}} \times (\mathbf{3} - \sqrt{\mathbf{3}}) \times \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} \times \mathbf{3} \times \mathbf{2} - \frac{\mathbf{1}}{\mathbf{2}} \times \sqrt{\mathbf{3}} \times \mathbf{3}$$

$$= \frac{\mathbf{9}}{\mathbf{2}} + \mathbf{2}\sqrt{\mathbf{3}}\ \boldsymbol{units}^{\mathbf{2}}$$