Section 2: The inverse hyperbolic functions

Notes and Examples

These notes contain subsections on

- **The logarithmic form of the inverse hyperbolic functions**
- **Differentiating the inverse hyperbolic functions**
- **Integration using the inverse hyperbolic functions**
- **Using hyperbolic substitutions**

The logarithmic form of the inverse hyperbolic functions

Unlike the circular functions, sine, cosine and tangent, and their inverses arcsin, arcos and arctan, you cannot find particular values of the hyperbolic functions and their inverses by pressing a button on a normal calculator. Many more advanced calculators do have the hyperbolic functions, but in an examination you will be expected to show how you have used the definitions of the hyperbolic and inverse hyperbolic functions.

The inverses of the hyperbolic functions are denoted by arsinh, arcosh and artanh (or sometimes by $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$).

Suppose you want to find arsinh 2. It is possible to solve an equation such as $\sinh x = 2$ by writing in terms of exponentials and rearranging to give a quadratic in e^x. This can be generalised as follows:

$$\sinh x = k$$

$$\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = k$$

$$\mathrm{e}^{2x} - 2k\mathrm{e}^x - 1 = 0$$

$$\mathrm{e}^x = \frac{2k \pm \sqrt{4k^2+4}}{2} = k \pm \sqrt{k^2+1}$$

$$x = \ln\left(k \pm \sqrt{k^2+1}\right)$$

Since $\sqrt{k^2+1} > k$, the negative square root can be discarded as this will not give a real root. So $x = \ln\left(k + \sqrt{k^2+1}\right)$, and this is the only real root of the equation $\sinh x = k$.

Therefore $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$.

The graph of $y = \sinh x$ shows that it is a one-to-one function, and so any equation of the form $\sinh x = k$ has one real root which can be found using the arsinh function.

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You can find the logarithmic forms of the arcosh and artanh functions in a similar way.

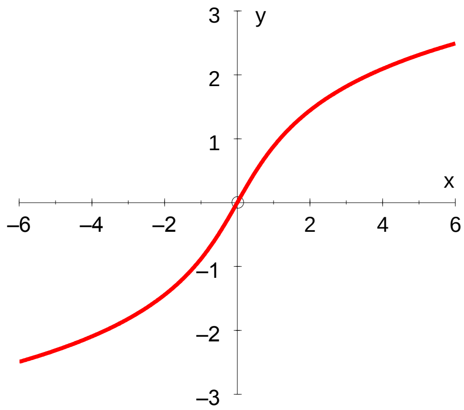
$$\text{arsinh}\, x = \ln(x + \sqrt{x^2 + 1})$$

$$\text{arcosh}\, x = \ln(x + \sqrt{x^2 - 1}),\ \ x \geq 1$$

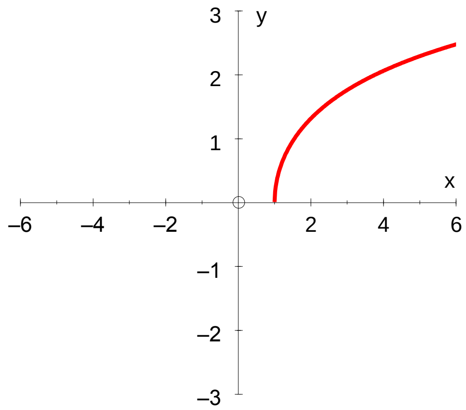
$$\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ \ |x| < 1$$

The reason for the restrictions on the values of x for arcosh and artanh can easily be seen by a glance at the graphs of the inverse hyperbolic functions. While arsinh is defined for all values of x, both arcosh and artanh have limited domains.

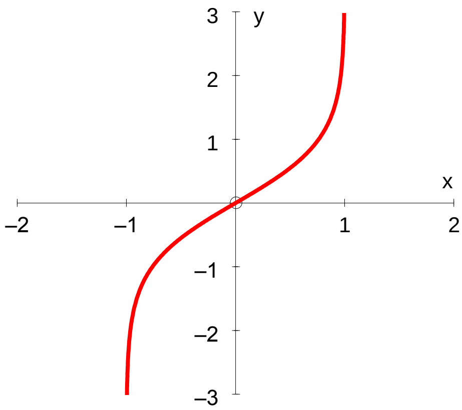
$y = \text{arsinh}\, x$



$y = \text{arcosh}\, x$



$y = \text{artanh} x$



These logarithmic forms for the inverse hyperbolic functions are given in your formula book.

It is important to notice that whereas the tanh function, like the sinh function, is one-to-one, the cosh function is not. This means that an equation of the form $\cosh x = k$ has two roots. The inverse cosh function $\text{arcosh}\, x = \ln(x + \sqrt{x^2 - 1})$ gives the positive root, and the other root is $-\ln(x + \sqrt{x^2 - 1})$. This is similar to solving a trigonometric equation like $\cos x = 0.4$ - your calculator gives you one root which is the value of arccos 0.4, but there are other roots which you can work out from the first root.

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Example 1 shows an equation that was solved in the Notes and Examples for section 1. In those notes, after the stage of reaching $\cosh x = \frac{3}{2}$, this was rewritten in terms of exponentials and solved. Here, the arcosh function is used, which saves time – but it is important not to forget the second root!

Example 1

Use the identitiy $\cosh 2x = 2\cosh^2 x - 1$ to solve the equation $\cosh 2x - \cosh x = 2$

Solution

$$\cosh 2x - \cosh x = 2$$
$$2\cosh^2 x - 1 - \cosh x = 2$$
$$2\cosh^2 x - \cosh x - 3 = 0$$
$$(2\cosh x - 3)(\cosh x + 1) = 0$$

Since $\cosh x$ cannot be negative, $\cosh x = \frac{3}{2}$

$$x = \operatorname{arcosh}\tfrac{3}{2} = \ln\left(\tfrac{3}{2} + \sqrt{(\tfrac{3}{2})^2 - 1}\right) = \ln\left(\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}\right)$$

By symmetry there is a second root which is the negative of this one.

$$x = \pm\ln\left[\tfrac{1}{2}\left(3 + \sqrt{5}\right)\right]$$

Notice that in the section 1 example, the roots were given as $x = \ln\left[\frac{1}{2}\left(3 \pm \sqrt{5}\right)\right]$. It may not be immediately obvious to you that $\ln\left(x - \sqrt{x^2 - 1}\right)$ is the same as $-\ln\left(x + \sqrt{x^2 - 1}\right)$ - you can prove this by writing $-\ln\left(x + \sqrt{x^2 - 1}\right)$ as $\ln\left(x + \sqrt{x^2 - 1}\right)^{-1}$ or $\ln\left(\dfrac{1}{x + \sqrt{x^2 - 1}}\right)$ and then rationalising the denominator.

Differentiating the inverse hyperbolic functions

The inverse hyperbolic functions can be differentiated using implicit differentiation, in the same way as for inverse trigonometric functions.

$$y = \operatorname{arsinh} x$$
$$\sinh y = x$$
$$\cosh y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

The derivatives of the inverse hyperbolic functions are as follows:

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$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

Integration using the inverse hyperbolic functions

The differentiation results above lead to some corresponding integration results which are given in your formula book.

$$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) \quad \text{or } \ln\left(x+\sqrt{x^2+a^2}\right)$$
$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) \quad \text{or } \ln\left(x+\sqrt{x^2-a^2}\right) \quad (x>a)$$

There is, of course, a corresponding result involving artanh; however this is the integral of $\frac{1}{a^2-x^2}$, which can be done using partial fractions, giving the same result.

These results can be quoted and applied directly.

Notice that the coefficient of x^2 must be 1 in all the results above. Sometimes expressions must be rewritten before the standard result can be applied.

Example 2

Find $\int \frac{1}{\sqrt{4x^2-1}}\,\mathrm{d}x$

Solution

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{4(x^2-\frac{1}{4})}}\,\mathrm{d}x \\ &= \int \frac{1}{2\sqrt{x^2-\frac{1}{4}}}\,\mathrm{d}x \\ &= \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\frac{1}{2}}\right)+c \\ &= \tfrac{1}{2}\operatorname{arcosh} 2x + c\end{aligned}$$

using the standard integral, with a = ½

Other, more complicated, functions can also be integrated using these standard results, by first completing the square. This is shown in the next example.

Example 3

Find $\displaystyle\int \frac{1}{\sqrt{4x^2+12x+5}}\,\mathrm{d}x$

Solution

$$\begin{aligned}4x^2+12x+5 &= 4\left(x^2+3x\right)+5\\ &= 4\left(x+\tfrac{3}{2}\right)^2-9+5\\ &= 4(x+\tfrac{3}{2})^2-4\end{aligned}$$

$$\begin{aligned}\int\frac{1}{\sqrt{4x^2+12x+5}}\,\mathrm{d}x &= \int\frac{1}{\sqrt{4\left(x+\tfrac{3}{2}\right)^2-4}}\,\mathrm{d}x\\ &= \int\frac{1}{2\sqrt{\left(x+\tfrac{3}{2}\right)^2-1}}\,\mathrm{d}x\\ &= \frac{1}{2}\operatorname{arcosh}\left(x+\tfrac{3}{2}\right)+c\end{aligned}$$

The introduction of the inverse hyperbolic functions in this section means that you can now integrate all functions of the forms

$$\frac{1}{Ax^2+Bx+C} \quad \text{and} \quad \frac{1}{\sqrt{Ax^2+Bx+C}}$$

(in the second case, on the condition that the function exists for some values of x!)

By taking out a factor A and completing the square, you can write any function of the form $\dfrac{1}{Ax^2+Bx+C}$ as a multiple of one of the following forms:

$$\frac{1}{(x-b)^2+a^2} \quad \text{or} \quad \frac{1}{(x-b)^2-a^2} \quad \text{or} \quad \frac{1}{a^2-(x-b)^2}$$

(You can also, of course, deal with $\dfrac{1}{-(x-b)^2-a^2}$ by writing it as $\dfrac{-1}{(x-b)^2+a^2}$).

All of these can now be integrated using standard integrals given in your formula book, or by standard methods, replacing x by $x-b$ where necessary:

$$\int\frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$

$\displaystyle\int\frac{1}{x^2-a^2}\,\mathrm{d}x$ and $\displaystyle\int\frac{1}{a^2-x^2}\,\mathrm{d}x$ can be integrating using partial fractions.

Similarly, by taking out a factor $\sqrt{A}$ and completing the square, you can write any function of the form $\dfrac{1}{\sqrt{Ax^2+Bx+C}}$ as a multiple of one of the following forms:

$$\frac{1}{\sqrt{(x-b)^2-a^2}} \quad \text{or} \quad \frac{1}{\sqrt{(x-b)^2+a^2}} \quad \text{or} \quad \frac{1}{\sqrt{a^2-(x-b)^2}}$$

As long as the function is defined for some values of x, these can all be integrated using standard integrals, replacing x by $x-b$ where necessary:

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right)+c \text{ or } \ln\left(x+\sqrt{x^2-a^2}\right)+c$$
$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right)+c \text{ or } \ln\left(x+\sqrt{x^2+a^2}\right)+c$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right)+c$$

Note: Clearly a function of the form $\frac{1}{\sqrt{-(x-b)^2-a^2}}$ is not real for any real values of x, so this one cannot be integrated!

In summary, you can now integrate any function of the form $\frac{1}{Ax^2+Bx+C}$ or $\frac{1}{\sqrt{Ax^2+Bx+C}}$ by the following method:

- If necessary, take out a factor so that the coefficient of x^2 is 1.
- If there is a term in x, write the expression in the completed square form.
- Select the appropriate standard integral from your formula book, or use partial fractions

You should get into the habit of working with your formula book in front of you, and looking up the integrals in that rather than in the textbook, so that you will be able to find them quickly in an examination.

Using hyperbolic substitutions

Just as trigonometric substitutions can sometimes be used to integrate expressions involving functions of $\sqrt{a^2-x^2}$ or a^2+x^2 (using the substitutions $x = a\sin u$ or $x = a\tan u$ respectively), expressions involving functions of $\sqrt{x^2-a^2}$ can be integrating using the substitution $x = a\cosh u$.

Example 4

Find $\int \sqrt{4x^2-1}\,\mathrm{d}x$

Solution

Let $x = \frac{1}{2}\cosh u \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}\sinh u$

$$\begin{aligned}
\int \sqrt{4x^2 - 1}\,\mathrm{d}x &= \int \sqrt{\cosh^2 u - 1} \times \tfrac{1}{2}\sinh u\,\mathrm{d}u \\
&= \int \sinh u \times \tfrac{1}{2}\sinh u\,\mathrm{d}u \\
&= \int \tfrac{1}{2}\sinh^2 u\,\mathrm{d}u \\
&= \int \tfrac{1}{2} \times \tfrac{1}{2}(\cosh 2u - 1)\,\mathrm{d}u \\
&= \tfrac{1}{4}(\tfrac{1}{2}\sinh 2u - u) + c \\
&= \tfrac{1}{4}\sinh u \cosh u - \tfrac{1}{4}u + c
\end{aligned}$$

$\cosh u = 2x \quad \Rightarrow \sinh u = \sqrt{(2x)^2 - 1} = \sqrt{4x^2 - 1}$

$$\begin{aligned}
\int \sqrt{4x^2 - 1}\,\mathrm{d}x &= \tfrac{1}{4}\sqrt{4x^2 - 1} \times 2x - \tfrac{1}{4}\operatorname{arcosh}(2x) + c \\
&= \tfrac{1}{2}x\sqrt{4x^2 - 1} - \tfrac{1}{4}\operatorname{arcosh}(2x) + c
\end{aligned}$$