# Section 2: The inverse hyperbolic functions

## Section test

**If you have a calculator with hyperbolic and inverse hyperbolic functions, do not use these functions for this test (except for checking). In an examination you will be expected to show that you have used the definitions of the hyperbolic and inverse hyperbolic functions where appropriate.**

1. Find the value of arsinh 2, correct to 2 decimal places.

2. Find the value of arcosh 3, correct to 2 decimal places.

3. The exact value of artanh $\frac{3}{5}$ is

(a) $-\ln 2$ (b) $\ln 4$
(c) $\frac{1}{2}\ln 2$ (d) $\ln 2$

4. Which of the following are roots of the equation $\cosh x = 2$?

$\ln\left(2+\sqrt{3}\right)$ $\ln\left(2-\sqrt{3}\right)$ $\ln\left(2+\sqrt{5}\right)$ $\ln\left(2-\sqrt{5}\right)$

5. The equation $\cosh x = 2\tanh x$ has one root. Find this root, correct to 3 s.f.

6. The derivative of arcosh $2x$ is

(a) $\frac{2}{\sqrt{4x^2-1}}$ (b) $\frac{1}{\sqrt{4x^2-1}}$
(c) $\frac{2}{\sqrt{2x^2-1}}$ (d) $\frac{1}{\sqrt{2x^2-1}}$

7. $\int \frac{1}{\sqrt{4x^2+9}}\,dx =$

(a) $\frac{1}{4}\text{arsinh}\frac{x}{3}+c$ (b) $\frac{1}{2}\text{arsinh}\frac{2x}{3}+c$
(c) $\frac{1}{2}\text{arsinh}\frac{x}{3}+c$ (d) $\frac{1}{4}\text{arsinh}\frac{2x}{3}+c$

8. $\int \frac{1}{\sqrt{x^2-2x-3}}\,dx =$

(a) $\text{arcosh}\left(\frac{x-1}{4}\right)+c$ (b) $\text{arcosh}\left(\frac{x-1}{2}\right)+c$
(c) $\text{arcosh}\left(\frac{x+1}{2}\right)+c$ (d) $\text{arcosh}\left(\frac{x+1}{4}\right)+c$

9. Find $\int_1^2 \frac{3}{\sqrt{9x^2-1}}\,dx$, correct to three significant figures.

10. Find $\int_0^1 \frac{2x+3}{\sqrt{x^2+4x+8}}\,dx$, correct to three significant figures.

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## Solutions to section test

1) $$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$
$$\operatorname{arsinh} 2 = \ln\left(2 + \sqrt{2^2 + 1}\right)$$
$$= \ln\left(2 + \sqrt{5}\right)$$
$$= 1.44$$

2) $$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$
$$\operatorname{arcosh} 3 = \ln\left(3 + \sqrt{3^2 - 1}\right)$$
$$= \ln\left(3 + \sqrt{8}\right)$$
$$= 1.76$$

3) $$\operatorname{artanh} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\operatorname{artanh}\tfrac{3}{5} = \tfrac{1}{2}\ln\left(\frac{1+\frac{3}{5}}{1-\frac{3}{5}}\right)$$
$$= \tfrac{1}{2}\ln\tfrac{8}{2}$$
$$= \tfrac{1}{2}\ln 4$$
$$= \ln 2$$

4) $$\operatorname{arcosh} 2 = \ln\left(2 + \sqrt{2^2 - 1}\right)$$
$$= \ln\left(2 + \sqrt{3}\right)$$

By symmetry the roots of $\cosh x = 2$ are $\ln\left(2+\sqrt{3}\right)$ and $-\ln\left(2+\sqrt{3}\right)$

$-\ln\left(2+\sqrt{3}\right)$ can be written as

$$\ln\left(2+\sqrt{3}\right)^{-1} = \ln\frac{1}{2+\sqrt{3}} = \ln\frac{(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \ln(2-\sqrt{3})$$

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5) $$\cosh x = 2\tanh x$$
$$\cosh x = \frac{2\sinh x}{\cosh x}$$
$$\cosh^2 x = 2\sinh x$$
$$1+\sinh^2 x = 2\sinh x$$
$$\sinh^2 x - 2\sinh x + 1 = 0$$
$$(\sinh x - 1)^2 = 0$$
$$\sinh x = 1$$
$$x = \ln\left(1+\sqrt{1^2+1}\right) = \ln(1+\sqrt{2})$$
$$= 0.881 \text{ (3 s.f.)}$$

6) $$\frac{d}{dx}(\operatorname{arcosh} 2x) = \frac{1}{\sqrt{(2x)^2-1}} \times 2$$
$$= \frac{2}{\sqrt{4x^2-1}}$$

7) $$\int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{2\sqrt{x^2+\frac{9}{4}}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx$$
$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{x}{\frac{3}{2}}\right) + c$$
$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + c$$

8) $$x^2 - 2x - 3 = (x-1)^2 - 1 - 3$$
$$= (x-1)^2 - 4$$
$$\int \frac{1}{\sqrt{x^2-2x-3}}\,dx = \int \frac{1}{\sqrt{(x-1)^2-4}}\,dx$$
$$= \operatorname{arcosh}\left(\frac{x-1}{2}\right) + c$$

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9) $$\int_1^2 \frac{3}{\sqrt{9x^2-1}}\,dx = \int_1^2 \frac{3}{3\sqrt{x^2-\frac{1}{9}}}\,dx = \int_1^2 \frac{1}{\sqrt{x^2-\frac{1}{9}}}\,dx$$

$$= \left[\ln\left(x+\sqrt{x^2-\tfrac{1}{9}}\right)\right]_1^2$$

$$= \ln\left(2+\sqrt{4-\tfrac{1}{9}}\right) - \ln\left(1+\sqrt{1-\tfrac{1}{9}}\right)$$

$$= \ln\left(\frac{2+\frac{1}{3}\sqrt{35}}{1+\frac{1}{3}\sqrt{8}}\right) = 0.715 \quad (3 \text{ s.f.})$$

10) $x^2+4x+8 = (x+2)^2-4+8 = (x+2)^2+4$

$$\int_0^1 \frac{2x+3}{\sqrt{x^2+4x+8}}\,dx = \int_0^1 \frac{2x+3}{\sqrt{(x+2)^2+4}}\,dx$$

$$= \int_0^1 \left(\frac{2x+4}{\sqrt{x^2+4x+8}} - \frac{1}{\sqrt{(x+2)^2+4}}\right)dx$$

$$= \left[2(x^2+4x+8)^{\frac{1}{2}} - \ln\left(x+2+\sqrt{(x+2)^2+4}\right)\right]_0^1$$

$$= 2\sqrt{13} - 2\sqrt{8} - \ln(3+\sqrt{13}) + \ln(2+\sqrt{8})$$

$$= 1.24 \quad (3 \text{ s.f.})$$