Section 1: Introducing the hyperbolic functions

Notes and Examples

These notes contain subsection on

The hyperbolic cosine and hyperbolic sine functions

In A level Mathematics, you learn that the unit circle $x^2 + y^2 = 1$ can be expressed in the parametric form $x = \cos\theta$, $y = \sin\theta$, where θ represents the angle between the x-axis and the line joining the point on the circle to the origin.

In a similar way, the hyperbola $x^2 - y^2 = 1$ can be expressed in parametric form $x = \cosh u$, $y = \sinh u$, where

$$\cosh u = \tfrac{1}{2}\left(e^{u} + e^{-u}\right) \quad \text{and} \quad \sinh u = \tfrac{1}{2}\left(e^{u} - e^{-u}\right)$$

These are called the hyperbolic functions.
There are many similarities between the properties of the circular functions, $\cos\theta$ and $\sin\theta$, and the properties of the hyperbolic functions, $\cosh u$ and $\sinh u$.

The hyperbolic function $\tanh x$ is given by $\tanh u = \dfrac{\sinh u}{\cosh u}$, so $\tanh u = \dfrac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$.

Example 1 below shows how the definitions of sinh and cosh can be used. Some more advanced calculators may be able to calculate hyperbolic functions and their inverses: however, although this is a useful check, in an examination you will be expected to show how you can use the definitions. Additionally, a calculator will only give you one answer, when there may be more than one.

In Example 1 two important techniques are used which are very commonly needed when working with hyperbolic functions. The first is that it is often helpful to multiply through by e^{x}, since this eliminates e^{-x} and gives an expression involving only positive indices. The second is dealing with a quadratic in e^{x}: remember that $e^{2x} = (e^{x})^2$, so an equation which involves terms in e^{2x}, e^{x} and a constant term is a quadratic equation and can be solved using the quadratic formula or by factorising.

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Example 1
Find the value(s) of x for which
(i) $\sinh x = -2$
(ii) $\cosh x = 3$

Solution
(i) $\sinh x = -2$

$$\frac{e^x - e^{-x}}{2} = -2$$

$$e^x - e^{-x} = -4$$

Multiply through by e^x

$$e^{2x} - 1 = -4e^x$$

$$e^{2x} + 4e^x - 1 = 0$$

This is a quadratic equation in e^x

Using the quadratic formula: $e^x = \dfrac{-4 \pm \sqrt{16+4}}{2} = \dfrac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$

$-2-\sqrt{5}$ is negative, and e^x cannot be negative, so there is only one solution.

$x = \ln\left(-2+\sqrt{5}\right)$

(ii) $\cosh x = 3$

$$\frac{e^x + e^{-x}}{2} = 3$$

$$e^x + e^{-x} = 6$$

Multiply through by e^x

$$e^{2x} + 1 = 6e^x$$

$$e^{2x} - 6e^x + 1 = 0$$

This is a quadratic equation in e^x

Using the quadratic formula: $e^x = \dfrac{6 \pm \sqrt{36-4}}{2} = \dfrac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$

Both of these are positive, so there are two solutions.

$x = \ln\left(3 \pm 2\sqrt{2}\right)$

Notice that in part (i) of Example 1 there was just one solution, but in part (ii) there were two. You could have predicted this by looking at the graphs of $\sinh x$ and $\cosh x$: $\sinh x$ is a one-to-one function, so there is just one value of x for every value of $\sinh x$. However, the graph of $y = \cosh x$ is an even function and has two values of x for every value of $\cosh x$ greater than 1.

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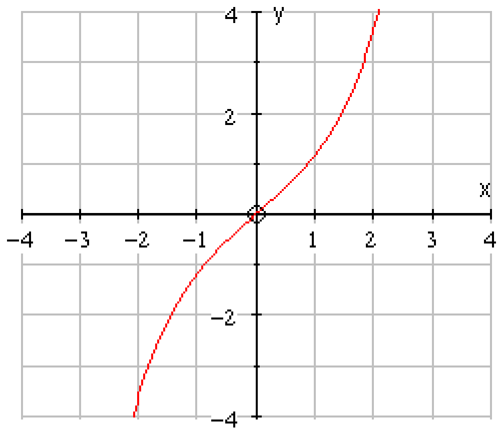
Graphs of the hyperbolic functions

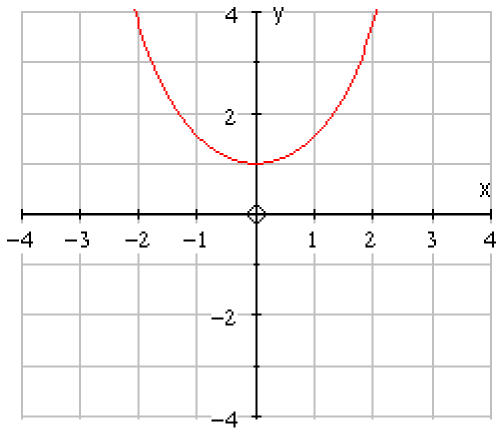
The graphs of the hyperbolic functions are given below. Note the domains and ranges of each.

$y = \sinh x$



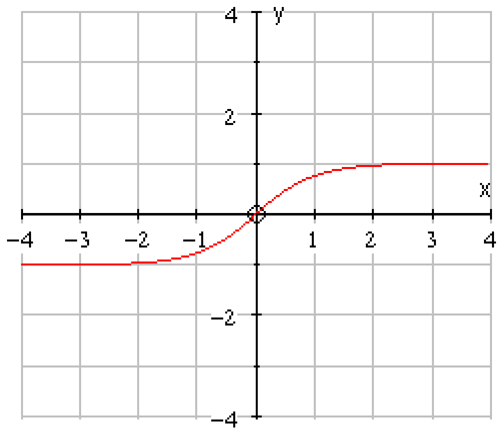
Domain: all real values of x
Range: all real values of y

$y = \cosh x$



Domain: all real values of x
Range: $y \geq 1$

$y = \tanh x$



Domain: all real values of x
Range: $-1 \leq y \leq 1$

Identities

The similarities between the properties of the hyperbolic sine and cosine functions and the circular sine and cosine functions mean that cosh and sinh are quite easy to deal with, as you can use many of the techniques that you are already familiar with, such as in solving equations, proving identities, differentiating and integrating.

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Thinking about a point $(\cos\theta, \sin\theta)$ as a general point on the unit circle $x^2 + y^2 = 1$ gives rise to the familiar identity $\cos^2 x + \sin^2 x \equiv 1$. In a similar way, thinking about a point $(\cosh\theta, \sinh\theta)$ as a point on the hyperbola $x^2 - y^2 = 1$ gives the identity

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

You should be familiar with this identity, and it is given in your formula book.
It is easy to prove using the definitions of the hyperbolic functions:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

There are other identities involving hyperbolic functions, which are similar to the corresponding trig identities.

The identities $\quad \sinh 2x = 2\sinh x \cosh x$
and $\quad \cosh 2x = \cosh^2 x + \sinh^2 x$
are given in your formula book.
Notice the similarity with the corresponding trigonometric identities, with just a sign difference in the second one. You can use the identity $\cosh^2 x - \sinh^2 x \equiv 1$ to express $\cosh 2x$ in terms of either $\sinh^2 x$ or $\cosh^2 x$.

You may be asked to prove hyperbolic identities by using the definitions.

Example 2 shows how hyperbolic identities can be used to solve equations in the same way that trig identities are used.

Example 2

(i) Prove the identity $\cosh 2x = 2\cosh^2 x - 1$

(ii) Solve the equation $\cosh 2x - \cosh x = 2$

Solution

(i)
$$\begin{aligned}
2\cosh^2 x - 1 &= 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 \\
&= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 \\
&= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x
\end{aligned}$$

(ii) $\cosh 2x - \cosh x = 2$

$2\cosh^2 x - 1 - \cosh x = 2$

$2\cosh^2 x - \cosh x - 3 = 0$

$(2\cosh x - 3)(\cosh x + 1) = 0$

Since $\cosh x$ cannot be negative, $\cosh x = \frac{3}{2}$

$\frac{1}{2}(e^x + e^{-x}) = \frac{3}{2}$

$e^x + e^{-x} = 3$

$e^{2x} + 1 = 3e^x$

$e^{2x} - 3e^x + 1 = 0$

Using the quadratic formula: $e^x = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

$x = \ln\left[\frac{1}{2}\left(3 \pm \sqrt{5}\right)\right]$

Differentiating and integrating hyperbolic functions

Using the definitions of the hyperbolic functions, it is easy to show that

$$\frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x$$

When differentiating and integrating, it is usually easiest to use the equivalent technique to the one you would use for trig functions. However, if you get stuck, you may find that going back to the definitions offers an alternative approach.

Example 3 demonstrates differentiating a variety of functions involving sinh and cosh. Notice how these are done in the same way as similar functions involving sine and cosine – in fact, they are easier, as no minus signs are involved in differentiating $\sinh x$ and $\cosh x$!

Example 3

Differentiate the following functions with respect to x.

(i) $\cosh 3x$

(ii) $\sinh^2 x$

(iii) $\cosh x \sinh 2x$

You can use the chain rule with $u = 3x$, but you should be able to write down a simple example like this one directly

Solution

(i) $\dfrac{d}{dx}(\cosh 3x) = 3\sinh 3x$

Use the chain rule, this time with $u = \sinh x$. Again, you can probably write it down directly

(ii) $\dfrac{d}{dx}(\sinh^2 x) = 2\sinh x \cosh x$

(iii) $u = \cosh x \Rightarrow \dfrac{du}{dx} = \sinh x$

$v = \sinh 2x \Rightarrow \dfrac{dv}{dx} = 2\cosh 2x$

Using the product rule

$$\frac{d}{dx}(\cosh x \sinh 2x) = \sinh x \sinh 2x + 2\cosh x \cosh 2x$$

Example 4 shows a variety of techniques of integration used to integrate functions involving sinh and cosh. Again, note the similarities with methods used for trig functions.

Example 4

Integrate the following functions with respect to x.

(i) $x\cosh 2x$

(ii) $\sinh^3 x$

(iii) $\cosh 3x \cosh 5x$

Solution

(i) $\int x\cosh 2x \, dx = \frac{1}{2}x\sinh 2x - \int \frac{1}{2}\sinh 2x \, dx$

$= \frac{1}{2}\sinh 2x - \frac{1}{4}\cosh 2x + c$

Here you need to use integration by parts

(ii) $\int \sinh^3 x dx = \int \sinh x(\cosh^2 x - 1)dx$

$= \int (\sinh x \cosh^2 x - \sinh x)dx$

$= \frac{1}{3}\cosh^3 x - \cosh x + c$

You need to use the identity $\cosh^2 x - \sinh^2 x = 1$

(iii) $\int \cosh 3x \cosh 5x dx = \int \frac{1}{2}\left(e^{3x} + e^{-3x}\right) \times \frac{1}{2}\left(e^{5x} + e^{-5x}\right) dx$

$= \int \frac{1}{4}\left(e^{8x} + e^{2x} + e^{-2x} + e^{-8x}\right) dx$

$= \frac{1}{2}\int \left(\frac{1}{2}\left(e^{8x} + e^{-8x}\right) + \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right) dx$

$= \frac{1}{2}\int \left(\cosh 8x + \cosh 2x\right) dx$

$= \frac{1}{16}\sinh 8x + \frac{1}{4}\sinh 2x + c$

You can of course integrate at this stage, and then convert back into cosh or sinh.

Example 5 shows definite integration. You need to be careful when substituting the limits.

Example 5

Find $\int_0^{\ln 2} (\cosh 2x - \sinh 3x) \, dx$.

Solution

$$\int_0^{\ln 2}(\cosh 2x-\sinh 3x)\,\mathrm{d}x=\left[\tfrac{1}{2}\sinh 2x-\tfrac{1}{3}\cosh 3x\right]_0^{\ln 2}$$

$$=\left[\tfrac{1}{4}\left(\mathrm{e}^{2x}-\mathrm{e}^{-2x}\right)-\tfrac{1}{6}\left(\mathrm{e}^{3x}+\mathrm{e}^{-3x}\right)\right]_0^{\ln 2}$$

$$=\tfrac{1}{4}\left(\mathrm{e}^{2\ln 2}-\mathrm{e}^{-2\ln 2}\right)-\tfrac{1}{6}\left(\mathrm{e}^{3\ln 2}+\mathrm{e}^{-3\ln 2}\right)-\tfrac{1}{4}(1-1)+\tfrac{1}{6}(1+1)$$

$$=\tfrac{1}{4}\left(\mathrm{e}^{\ln 4}-\mathrm{e}^{\ln\frac{1}{4}}\right)-\tfrac{1}{6}\left(\mathrm{e}^{\ln 8}+\mathrm{e}^{\ln\frac{1}{8}}\right)+\tfrac{1}{6}\times 2$$

$$=\tfrac{1}{4}\left(4-\tfrac{1}{4}\right)-\tfrac{1}{6}\left(8+\tfrac{1}{8}\right)+\tfrac{1}{3}$$

$$=\tfrac{15}{16}-\tfrac{65}{48}+\tfrac{1}{3}$$

$$=-\tfrac{1}{12}$$

Remember $2\ln 2=\ln 2^2$ etc, and that $\mathrm{e}^{\ln x}=x$

Power series for hyperbolic functions

You have learned about the Maclaurin series for common functions such as e^x, $\sin x$ and $\cos x$. You can now add the Maclaurin series for $\cosh x$ and $\sinh x$.

Recall the general Maclaurin series for a function $\mathrm{f}(x)$:

$$\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\frac{x^3}{3!}\mathrm{f}^{(3)}(0)+\ldots+\frac{x^r}{r!}\mathrm{f}^{(r)}(0)+\ldots$$

For

$$\begin{array}{lll}
\mathrm{f}(x)=\cosh x & \mathrm{f}(0)=1 & \\
\mathrm{f}'(x)=\sinh x & \mathrm{f}'(0)=0 & \\
\mathrm{f}''(x)=\cosh x & \mathrm{f}''(0)=1 & \\
\mathrm{f}^{(3)}(x)=\sinh x & \mathrm{f}^{(3)}(0)=0 & \\
\mathrm{f}^{(4)}(x)=\cosh x & \mathrm{f}^{(4)}(0)=1 & \text{etc.}
\end{array}$$

So $$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2r}}{(2r)!}+\ldots$$

You can find the Maclaurin series for $\sinh x$ in a similar way:

$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{x^{2r+1}}{(2r+1)!}+\ldots$$

Notice that these series for $\cosh x$ and $\sinh x$ are the same as those for $\cos x$ and $\sin x$ respectively except that all signs are positive.
Alternatively, you can find these Maclaurin series for by using the definitions of $\cosh x$ and $\sinh x$ and the known Maclaurin series for e^x:

$$
\begin{aligned}
\cosh x &= \frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)\\
&= \frac{1}{2}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}...+\frac{x^r}{r!}+...+1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+...+\frac{(-x)^r}{r!}+...\right)\\
&= \frac{1}{2}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}...+\frac{x^r}{r!}+...+1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+...+\frac{(-1)^r x^r}{r!}+...\right)\\
&= \frac{1}{2}\left(2+\frac{2x^2}{2!}+\frac{2x^4}{4!}+...\frac{2x^{2r}}{(2r)!}+...\right)\\
&= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+...\frac{x^{2r}}{(2r)!}+...
\end{aligned}
$$

All the terms in odd powers of x cancel out

Similarly for $\sinh x$ – in this case all the terms in even powers of x cancel out.
Both these series are valid for all values of x.