Section 1: The method of differences

Section test

1. Write the expression $\dfrac{1}{r(r+2)}$ in partial fractions.

2. Using the result from the previous question, the sum $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+2)}$ is given by

(a) $\dfrac{4n+9}{4(n+1)(n+2)}$

(b) $\dfrac{n+1}{2(n+2)}$

(c) $\dfrac{n(3n+5)}{4(n+1)(n+2)}$

(d) $\dfrac{n+3}{2(n+2)}$

3. By writing $\dfrac{1}{(2r-1)(2r+1)}$ in partial fractions, find $\displaystyle\sum_{r=1}^{100}\frac{1}{(2r-1)(2r+1)}$

4. Using the result $3r^2+3r+1=(r+1)^3-r^3$, find the sum $\displaystyle\sum_{r=1}^{n}\frac{3r^2+3r+1}{(r+1)^3r^3}$ and hence evaluate $\displaystyle\sum_{r=1}^{20}\frac{3r^2+3r+1}{(r+1)^3r^3}$.

5. Write $\dfrac{8}{(2r-1)(2r+1)(2r+3)}$ in partial fractions.

6. Using the result from the previous question, the sum

$$\frac{8}{1\times3\times5}+\frac{8}{3\times5\times7}+\frac{8}{5\times7\times9}+\ldots+\frac{8}{(2n-1)(2n+1)(2n+3)}$$

is given by

(a) $1-\dfrac{1}{2n+3}$

(b) $1+\dfrac{1}{2n+3}$

(c) $\dfrac{1}{3}-\dfrac{2}{2n+1}+\dfrac{1}{2n+3}$

(d) $\dfrac{2}{3}-\dfrac{1}{2n+1}+\dfrac{1}{2n+3}$

7. Write the expression $\dfrac{r}{(r+2)(r+3)(r+4)}$ in partial fractions.

Hence find the value of the sum

$$\frac{1}{3\times4\times5}+\frac{2}{4\times5\times6}+\frac{3}{5\times6\times7}+\ldots+\frac{12}{14\times15\times16}$$

Edexcel Sequences and series 1 Section test solutions

Solutions to section test

1. $$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$
$$1 = A(r+2) + Br$$
Putting $r = 0 \Rightarrow 1 = 2A \quad \Rightarrow A = \tfrac{1}{2}$
Putting $r = -2 \Rightarrow 1 = -2B \quad \Rightarrow B = -\tfrac{1}{2}$
$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

2. $$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{2(r+2)}\right)$$
$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \ldots$$
$$\ldots + \left(\frac{1}{2(n-1)} - \frac{1}{2(n+1)}\right) + \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right)$$
$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$
$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$
$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$
$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$
$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$

3. $$\frac{1}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$
$$1 = A(2r+1) + B(2r-1)$$
Putting $r = -\tfrac{1}{2} \quad \Rightarrow 1 = -2B \quad \Rightarrow B = -\tfrac{1}{2}$
Putting $r = \tfrac{1}{2} \quad \Rightarrow 1 = 2A \quad \Rightarrow A = \tfrac{1}{2}$
$$\frac{1}{(2r-1)(2r+1)} = \frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$$
$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n}\left(\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}\right)$$
$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \ldots + \left(\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}\right)$$
$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$

$$\text{So } \sum_{r=1}^{100} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2} - \frac{1}{2(200+1)}$$
$$= \frac{1}{2} - \frac{1}{402}$$
$$= \frac{100}{201}$$

4. $$\sum_{r=1}^{n} \frac{3r^2+3r+1}{(r+1)^3 r^3} = \sum_{r=1}^{n} \frac{(r+1)^3 - r^3}{(r+1)^3 r^3}$$
$$= \sum_{r=1}^{n} \left(\frac{1}{r^3} - \frac{1}{(r+1)^3} \right)$$
$$= \left(1 - \frac{1}{2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{3^3}\right) + \ldots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right)$$
$$= 1 - \frac{1}{(n+1)^3}$$

$$\sum_{r=1}^{20} \frac{3r^2+3r+1}{(r+1)^3 r^3} = 1 - \frac{1}{21^3}$$
$$= \frac{9260}{9261}$$

5. $$\frac{8}{(2r-1)(2r+1)(2r+3)} = \frac{A}{2r-1} + \frac{B}{2r+1} + \frac{C}{2r+3}$$
$$8 = A(2r+1)(2r+3) + B(2r-1)(2r+3) + C(2r-1)(2r+1)$$
Putting $r = \frac{1}{2} \quad \Rightarrow 8 = 2 \times 4A \quad \Rightarrow A = 1$

Putting $r = -\frac{1}{2} \quad \Rightarrow 8 = -2 \times 2B \quad \Rightarrow B = -2$

Putting $r = -\frac{3}{2} \quad \Rightarrow 8 = -4 \times -2C \quad \Rightarrow C = 1$

$$\frac{8}{(2r-1)(2r+1)(2r+3)} = \frac{1}{2r-1} - \frac{2}{2r+1} + \frac{1}{2r+3}$$

Edexcel Sequences and series 1 Section test solutions

6. $$\sum_{r=1}^{n}\frac{8}{(2r-1)(2r+1)(2r+3)}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{2}{2r+1}+\frac{1}{2r+3}\right)$$

$$=\left(1-\frac{2}{3}+\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)$$

$$+\ldots+\left(\frac{1}{2n-3}-\frac{2}{2n-1}+\frac{1}{2n+1}\right)$$

$$+\left(\frac{1}{2n-1}-\frac{2}{2n+1}+\frac{1}{2n+3}\right)$$

$$=1-\frac{2}{3}+\frac{1}{3}+\frac{1}{2n+1}-\frac{2}{2n+1}+\frac{1}{2n+3}$$

$$=\frac{2}{3}-\frac{1}{2n+1}+\frac{1}{2n+3}$$

7. $$\frac{r}{(r+2)(r+3)(r+4)}=\frac{A}{r+2}+\frac{B}{r+3}+\frac{C}{r+4}$$

$r=A(r+3)(r+4)+B(r+2)(r+4)+C(r+2)(r+3)$

Putting r = -2 $\Rightarrow -2=2A \quad \Rightarrow A=-1$

Putting r = -3 $\Rightarrow -3=-B \quad \Rightarrow B=3$

Putting r = -4 $\Rightarrow -4=2C \quad \Rightarrow C=-2$

$$\frac{r}{(r+2)(r+3)(r+4)}=-\frac{1}{r+2}+\frac{3}{r+3}-\frac{2}{r+4}$$

$$\frac{1}{3\times4\times5}+\frac{2}{4\times5\times6}+\frac{3}{5\times6\times7}+\ldots+\frac{12}{14\times15\times16}$$

$$=\left(-\frac{1}{3}+\frac{3}{4}-\frac{2}{5}\right)+\left(-\frac{1}{4}+\frac{3}{5}-\frac{2}{6}\right)+\left(-\frac{1}{5}+\frac{3}{6}-\frac{2}{7}\right)+\ldots$$

$$\ldots+\left(-\frac{1}{13}+\frac{3}{14}-\frac{2}{15}\right)+\left(-\frac{1}{14}+\frac{3}{15}-\frac{2}{16}\right)$$

$$=-\frac{1}{3}+\frac{3}{4}-\frac{1}{4}-\frac{2}{15}+\frac{3}{15}-\frac{2}{16}$$

$$=\frac{13}{120}$$