Section 1: The method of differences

Notes and Examples

These notes contain subsections on
- The method of differences
- Using partial fractions

The method of differences

In the method of differences, a series is expressed as the difference of two (or more) other sequences, so that most of the terms cancel out.

Not all series can be summed in this way. The main difficulty in this method is deciding whether it can be used, and seeing how to express the series so that terms do cancel out.

Example 1

(i) Simplify $r(r+1)(r+2)-(r-1)r(r+1)$.

(ii) Hence find $\sum_{r=1}^{n} r(r+1)$

Solution

(i)
$$\begin{aligned} r(r+1)(r+2)-(r-1)r(r+1) &= r^3+3r^2+2r-(r^3-r) \\ &= 3r^2+3r \\ &= 3r(r+1) \end{aligned}$$

(ii)
$$\therefore\ r(r+1) = \tfrac{1}{3}\big(r(r+1)(r+2)-(r-1)r(r+1)\big)$$
$$\begin{aligned} \sum_{r=1}^{n} r(r+1) &= \tfrac{1}{3}\sum_{r=1}^{n}\big(r(r+1)(r+2)-(r-1)r(r+1)\big) \\ &= \tfrac{1}{3}\Big[(\cancel{1\times2\times3}-0)+(\cancel{2\times3\times4}-\cancel{1\times2\times3})+(\cancel{3\times4\times5}-\cancel{2\times3\times4}) \\ &\qquad +\dots\dots+n(n+1)(n+2)-\cancel{(n-1)n(n+1)}\Big] \end{aligned}$$

Virtually all these terms cancel in pairs, leaving just two terms, one of which is zero in this case.

$$\begin{aligned} \sum_{r=1}^{n} r(r+1) &= \tfrac{1}{3}n(n+1)(n+2)-0 \\ &= \tfrac{1}{3}n(n+1)(n+2) \end{aligned}$$

Edexcel Sequences and series 1 Notes and Examples

Using partial fractions

If the terms to be summed include fractions, such as $\sum_{r=1}^{n} \frac{2}{4r^2-1}$, you may need to use **partial fractions.**

Example 2

Find the value of $\sum_{r=1}^{n} \frac{2}{4r^2-1}$.

Solution

$$\frac{2}{4r^2-1} = \frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

$$2 = A(2r+1) + B(2r-1)$$

Putting $r = -\frac{1}{2} \quad \Rightarrow 2 = -2B \quad \Rightarrow B = -1$

Putting $r = \frac{1}{2} \quad \Rightarrow 2 = 2A \quad \Rightarrow A = 1$

$$\frac{2}{4r^2-1} = \frac{1}{2r-1} - \frac{1}{2r+1}$$

$$\begin{aligned}\sum_{r=1}^{n} \frac{2}{4r^2-1} &= \sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) \\ &= \left\{\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots + \cdots \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right\} \\ &= 1 - \frac{1}{2n+1} \\ &= \frac{2n}{2n+1}\end{aligned}$$

Most of the terms cancel in pairs

When using the method of differences, you need to make sure that you have correctly identified the terms which are left over, after cancelling out other terms. In some examples this is very easy and there are just two terms left over. However, there are other possibilities – there may be two terms left over at the beginning and two at the end, for example. Here is a further example in which there are three terms for each value of *r*.

Example 5

(i) Express $\frac{r+4}{r(r+1)(r+2)}$ in partial fractions.

(ii) Hence find $\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)}$

Solution

(i) $$\frac{r+4}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$r+4 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$$

Putting $r = 0 \quad \Rightarrow 4 = 1 \times 2A \quad \Rightarrow A = 2$

Putting $r = -1 \quad \Rightarrow 3 = -1 \times 1B \quad \Rightarrow B = -3$

Putting $r = -2 \quad \Rightarrow 2 = -2 \times -1C \quad \Rightarrow C = 1$

$$\frac{r+4}{r(r+1)(r+2)} = \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2}$$

(ii) $$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \right)$$

It is best not to simplify fractions

$r = 1 \qquad 2 - \frac{3}{2} + \cancel{\frac{1}{3}}$

$r = 2 \qquad \frac{2}{2} - \cancel{\frac{3}{3}} + \cancel{\frac{1}{4}}$

$r = 3 \qquad \cancel{\frac{2}{3}} - \cancel{\frac{3}{4}} + \cancel{\frac{1}{5}}$

$r = 4 \qquad \cancel{\frac{2}{4}} - \cancel{\frac{3}{5}} + \cancel{\frac{1}{6}}$

You can see how the thirds and quarters cancel

Fifths, sixths and so on will cancel with terms from the following rows

…. ……

$r = n-2 \qquad \cancel{\frac{2}{n-2}} - \cancel{\frac{3}{n-1}} + \cancel{\frac{1}{n}}$

$r = n-1 \qquad \cancel{\frac{2}{n-1}} - \cancel{\frac{3}{n}} + \frac{1}{n+1}$

$r = n \qquad \cancel{\frac{2}{n}} - \frac{3}{n+1} + \frac{1}{n+2}$

You can see that terms with denominator n cancel. Terms with denominators $n - 1$ and $n - 2$ also cancel with terms in previous rows.

There are six terms remaining: three at the beginning (one with denominator 1 and two with denominator 2) and three at the end (two with denominator $n - 1$ and one with denominator $n + 2$).

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = 2 - \frac{3}{2} + \frac{2}{2} + \frac{1}{n+1} - \frac{3}{n+1} + \frac{1}{n+2}$$

$$= \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$$