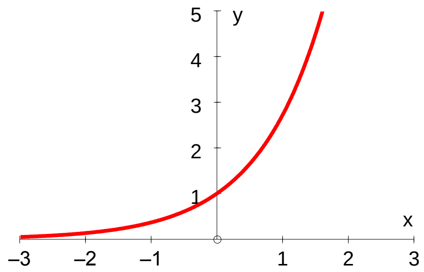
Section 1: Finding and using Maclaurin series

Notes and Examples

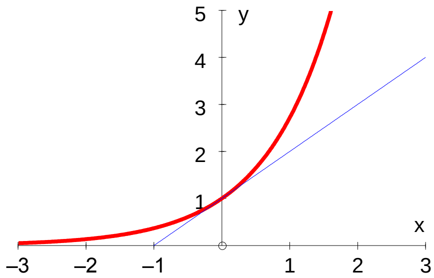
These notes contain subsections on

Polynomial approximations

Consider the function $y = \mathrm{e}^x$.



If you want a straight line approximation to the graph at the point where $x = 0$, the most sensible choice is the tangent at $x = 0$. This is the line $y = 1 + x$. It goes through the point (0, 1), and the gradient of this line is the same as the gradient of $y = \mathrm{e}^x$ at (0, 1).



However, a straight line is not a very good approximation to a curve, particularly further away from (0, 1). A quadratic equation would be a better approximation. To find a quadratic $y = a + bx + cx^2$ which is a good approximation to $y = \mathrm{e}^x$, we need three pieces of information to find the values of a, b and c. As for the straight line, it should go through the point (0, 1) and it should have the same gradient at (0, 1). In addition, it should have the same second derivative at (0, 1). The quadratic function which fulfils all these requirements is $y = 1 + x + \frac{1}{2}x^2$.

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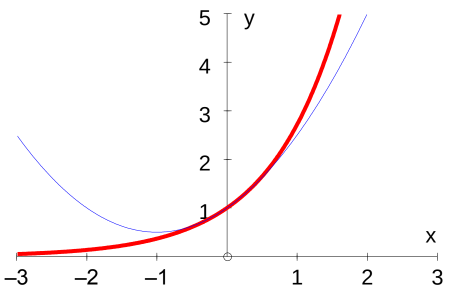
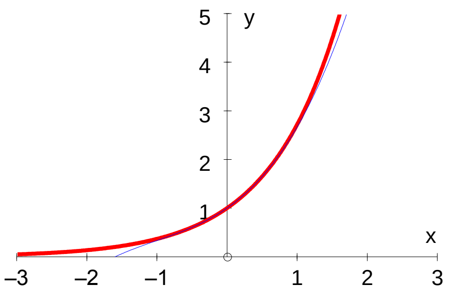


You can see that this is a better approximation to $y = e^x$. The approximation is good for a wider range of values of x, although it is still not good at points further from $x = 0$.

A cubic function will be an even better approximation. This must pass through (0, 1) and have the same gradient and the same value for the second and third derivatives at (0, 1) as $y = e^x$. The cubic function which fulfils all these requirements is $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.



This is a better approximation, and again it is good for a wider range of values of x.

You can continue this process indefinitely.

The general Maclaurin expansion

The ideas above can be generalised for any function, so long as the function and its derivatives exist at $x = 0$.

The Maclaurin series for a function $f(x)$ for which f and its derivatives exist at $x = 0$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Example 1
Obtain the first two non-zero terms in the power series expansion of $\arctan x$.

Solution

$$\mathrm{f}(x) = \mathrm{f}(0) + \frac{\mathrm{f}'(0)}{1!}x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}'''(0)}{3!}x^3 + ...$$

$\mathrm{f}(x) = \arctan x \qquad \mathrm{f}(0) = 0$

$\mathrm{f}'(x) = \dfrac{1}{1+x^2} \qquad \mathrm{f}'(0) = 1$

$\mathrm{f}''(x) = -\dfrac{2x}{(1+x^2)^2} \qquad \mathrm{f}''(0) = 0$

$\mathrm{f}'''(x) = \dfrac{-2(1+x^2)^2 + 2x \times 2(1+x^2) \times 2x}{(1+x^2)^4} \qquad \mathrm{f}'''(0) = -2$

$$\mathrm{f}(x) = \frac{1}{1!}x - \frac{2}{3!}x^3 + ...$$

$$\mathrm{f}(x) = x - \frac{1}{3}x^3 + ...$$

Standard Maclaurin series

The Maclaurin expansions for certain common functions can be quoted unless you are asked to derive them.

Some of the standard Maclaurin series which you may quote are given below, together with the range of values of x for which they are valid. (The explanation of why some power series expansions are only valid for certain values of x is beyond the scope of this course, but some justification is given after Example 3).

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + ... + \dfrac{x^r}{r!} + ...$	Valid for all x
$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - .... + (-1)^r \dfrac{x^{2r+1}}{(2r+1)!} + ...$	Valid for all x
$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - ... + (-1)^r \dfrac{x^{2r}}{(2r)!} + ...$	Valid for all x
$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - ... + (-1)^{r+1}\dfrac{x^r}{r} + ...$	Valid for $-1 < x \leq 1$

Note that:

- for the trigonometric functions, x must be in radians, since you can only differentiate trig functions when working in radians.
- you cannot find a Maclaurin expansion for $\ln x$, since $\ln x$ does not exist for $x = 0$.
- if you differentiate the power series for $\sin x$ term by term, you get the power series for $\cos x$, and if you differentiate the power series for $\cos x$, you get the power series for $-\sin x$, just as you would expect.

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The power series for standard functions can be used to find other power series for functions based on these, as shown in the next example.

Example 2

Obtain the first four non-zero terms in the power series expansions of

(i) $\sin\left(\frac{1}{2}x\right)$

(ii) e^{-3x}

(iii) $\ln(1+2x)$

giving the range of validity of each series.

Solution

(i) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots\dots$

$$\sin\left(\tfrac{1}{2}x\right) = \tfrac{1}{2}x - \frac{\left(\frac{1}{2}x\right)^3}{3!} + \frac{\left(\frac{1}{2}x\right)^5}{5!} - \frac{\left(\frac{1}{2}x\right)^7}{7!} + \dots\dots$$

$$= \frac{x}{2} - \frac{x^3}{2^3 \times 3!} + \frac{x^5}{2^5 \times 5!} - \frac{x^7}{2^7 \times 7!} + \dots\dots$$

$$= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645120} + \dots\dots$$

Valid for all values of x.

(ii) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots\dots$

$$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \dots\dots$$

$$= 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \dots$$

$$= 1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \dots$$

Valid for all values of x.

(iii) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots\dots$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots\dots$$

$$= 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \dots\dots$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots\dots$$

Valid for $-1 < 2x \le 1$

i.e. $-\tfrac{1}{2} < x \le \tfrac{1}{2}$

Be careful about the range of validity!

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Be careful to make sure that it is appropriate to use a standard Maclaurin expansion. Suppose, for example, that you want to find the expansion for $\ln(1+\cos x)$.

You might start by substituting $\cos x$ into the expansion for $\ln(1+x)$ which would give $\cos x - \frac{\cos^2 x}{2} + \frac{\cos^3 x}{3} + \ldots$

However, think about what would happen if you then use the expansion for $\cos x$ in each term. Every power of $\cos x$ would give a constant term, a term in x, a term in x^2 and so on – so you would need to use the terms in $\cos^4 x$ and all higher terms to get all the constant terms, all the terms in x and so on. So in cases like these, you will need to use repeated differentiation and the standard Maclaurin expansion, as shown in Example 1.

Using a Maclaurin series to find an approximate value

The example below shows how you can use the first few terms of a Maclaurin series to find an approximate value.

Example 3

(i) Use the first four terms of the power series for e^x to find an approximate value for $e^{0.5}$. Find the percentage error in this value.

(ii) Use the first four terms of the power series for e^x to find an approximate value for e^2. Find the percentage error in this value.

(iii) Use the first eight terms of the power series for e^x to find an approximate value for e^2. Find the percentage error in this value.

Solution

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$

$$e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} + \ldots. = 1.645833333$$

Using a calculator $e^{0.5} = 1.648721271$

Percentage error $= \frac{(1.648721271 - 1.645833333)}{1.648721271} \times 100 = 0.175\%$

(ii) $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots. = 6.333333333$

Using a calculator $e^2 = 7.389056099$

Percentage error $= \frac{(7.389056099 - 6.333333333)}{7.389056099} \times 100 = 14.3\%$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \ldots.$

$$e^x = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} \ldots. = 7.380952381$$

Using a calculator $e^2 = 7.389056099$

Percentage error $= \dfrac{(7.389056099 - 7.380952381)}{7.389056099} \times 100 = 0.11\%$

Example 3 shows that for a small value of x, using the first few terms of the power series for e^x provides a good approximation. For larger values of x, more terms are needed for the approximation to be reasonable. This is because for large values of x, the first few terms in the series increase, but eventually, however large x is, the factorial in the denominator will become larger than the power of x in the numerator, so that the terms will start to become smaller and the series will start to converge.

If you want to find a value correct to a specified degree of accuracy, you need to use enough terms so that adding the next term does not affect your answer to the degree of accuracy that you need.

Example 4

Use the Maclaurin series for cos x to find the value of cos 0.5 correct to 5 decimal places.

Solution

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$

Using the first two terms $\quad \cos 0.5 \approx 1 - \dfrac{0.5^2}{2!} = 0.875$

Using the first three terms $\quad \cos 0.5 \approx 1 - \dfrac{0.5^2}{2!} + \dfrac{0.5^4}{4!} = 0.87760417$

Using the first four terms $\quad \cos 0.5 \approx 1 - \dfrac{0.5^2}{2!} + \dfrac{0.5^4}{4!} - \dfrac{0.5^6}{6!} = 0.87758247$

Using the first five terms $\quad \cos 0.5 \approx 1 - \dfrac{0.5^2}{2!} + \dfrac{0.5^4}{4!} - \dfrac{0.5^6}{6!} + \dfrac{0.5^8}{8!} = 0.87758256$

Adding the fifth term does not affect the fifth decimal place.

so cos 0.5 = 0.87758 correct to 5 decimal places.