Section 2: Applications of de Moivre's theorem

Section test

1) The complex number -2 + 2i can be written in exponential form as

(a) $2e^{\frac{3i\pi}{4}}$

(b) $2\sqrt{2}e^{\frac{3i\pi}{4}}$

(c) $2\sqrt{2}e^{-\frac{i\pi}{4}}$

(d) $2e^{-\frac{i\pi}{4}}$

2) The complex number $2e^{\frac{2i\pi}{3}}$ can be written in Cartesian form as

(a) $-1+\sqrt{3}i$

(b) $-\sqrt{3}+i$

(c) $1-\sqrt{3}i$

(d) $\sqrt{3}-i$

3) The complex number $-4\sqrt{3}+4i$ can be expressed in the form $re^{i\theta}$ as

(a) $8e^{\frac{5i\pi}{6}}$

(b) $4e^{\frac{5i\pi}{6}}$

(c) $4e^{-\frac{i\pi}{6}}$

(d) $8e^{-\frac{i\pi}{6}}$

4) $e^{2i\theta} = R(\cos\alpha + i\sin\alpha)$, find the value of R and express α in terms of θ.

5) If $z = \cos\theta + i\sin\theta$, the value of $z^3 + \frac{1}{z^3}$ is

(a) $2i\sin 3\theta$

(b) $2\cos 3\theta$

(c) $i\sin 3\theta$

(d) $\cos 3\theta$

6) $\cos 4\theta$ is equivalent to

(a) $8\cos^4\theta + 8\cos^2\theta + 1$

(b) $8\cos^4\theta - 8\cos^2\theta + 1$

(c) $8\cos^4\theta - 8\cos^2\theta - 1$

(d) $8\cos^4\theta + 8\cos^2\theta - 1$

7) $\sin 4\theta$ is equivalent to

(a) $8\cos^3\theta\sin\theta - 4\cos\theta\sin\theta$

(b) $4\cos^3\theta\sin\theta + 4\cos\theta\sin\theta$

(c) $4\cos^3\theta\sin\theta - 4\cos\theta\sin\theta$

(d) $8\cos^3\theta\sin\theta + 8\cos\theta\sin\theta$

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8) Which of the following equals $\cos^6\theta$?

(a) $\dfrac{3\cos 6\theta + 12\cos 4\theta + 25\cos 2\theta + 24}{64}$

(b) $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$

(c) $\dfrac{2\cos 6\theta + 4\cos 4\theta + 10\cos 2\theta + 16}{32}$

(d) $\dfrac{17\cos 6\theta + 15\cos 4\theta + 17\cos 2\theta + 15}{32}$

9) If $C = 1 + {}_3C_1\cos\theta + {}_3C_2\cos 2\theta + {}_3C_3\cos 3\theta$ and $S = {}_3C_1\sin\theta + {}_3C_2\sin 2\theta + {}_3C_3\sin 3\theta$, then $C + \mathrm{i}S$ equals which of the following?

(a) $1 + \left(\mathrm{e}^{\mathrm{i}\theta}\right)^3$

(b) $\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)^4$

(c) $\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)^3$

(d) $1 + \left(\mathrm{e}^{\mathrm{i}\theta}\right)^4$

10) Let $S = \dfrac{\sin\theta}{2} - \dfrac{\sin 2\theta}{2^2} + \dfrac{\sin 3\theta}{2^3} - \dfrac{\sin 4\theta}{2^4} + ...$ and let

$C = 1 - \dfrac{\cos\theta}{2} + \dfrac{\cos 2\theta}{2^2} - \dfrac{\cos 3\theta}{2^3} + \dfrac{\cos 4\theta}{2^4} - ...$.

Then $C - \mathrm{i}S$ equals which of the following?

(a) $\dfrac{1}{2} + \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2} - \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^2 + \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^3 - ...$

(b) $\dfrac{1}{2} - \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2} + \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^2 - \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^3 + ...$

(c) $1 + \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2} - \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^2 + \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^3 - ...$

(d) $1 - \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2} + \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^2 - \left(\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{2}\right)^3 + ...$

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Solutions to section test

1. The modulus of -2 + 2i is $\sqrt{2^2+2^2}=2\sqrt{2}$

   Since -2 + 2i is in the second quadrant, the argument of -2 + 2i is given by

   $$\pi+\tan^{-1}\left(\frac{2}{-2}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}.$$

   So the exponential form is $2\sqrt{2}e^{\frac{3i\pi}{4}}$.

2. $$2e^{\frac{2i\pi}{3}}=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$
   $$=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$
   $$=-1+\sqrt{3}i$$

3. The modulus of $-4\sqrt{3}+4i$ is $\sqrt{48+16}=\sqrt{64}=8$

   Since $-4\sqrt{3}+4i$ is in the second quadrant, the argument of $-4\sqrt{3}+4i$ is given by

   $$\pi+\tan^{-1}\left(\frac{4}{-4\sqrt{3}}\right)=\pi+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}.$$

   So the exponential form is $8e^{\frac{5i\pi}{6}}$.

4. $e^{2i\theta}=\cos 2\theta+i\sin 2\theta$

   so $R=1$ and $\alpha=2\theta$

5. $z^3=\cos 3\theta+i\sin 3\theta$

   $z^{-3}=\cos 3\theta-i\sin 3\theta$

   Adding: $z^3+z^{-3}=2\cos 3\theta$

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6. $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

$$\cos 4\theta = \mathrm{Re}(\cos 4\theta + i\sin\theta)^4$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

7. $\sin 4\theta = \mathrm{Im}(\cos\theta + i\sin\theta)^4$

$$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

$$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin\theta(1-\sin^2\theta)$$

$$= 8\cos^3\theta\sin\theta - 4\cos\theta\sin\theta$$

8. $\cos\theta = \dfrac{z + z^{-1}}{2}$

$$\cos^6\theta = \frac{(z+z^{-1})^6}{2^6}$$

$$= \frac{z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6zz^{-5} + z^{-6}}{64}$$

$$= \frac{z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}}{64}$$

$$\frac{z^6 + z^{-6} + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20}{64}$$

$$= \frac{2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20}{64}$$

$$= \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$$

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9. $C + iS = 1 + {}_3C_1\cos\theta + {}_3C_2\cos 2\theta + {}_3C_3\cos 3\theta + i\left({}_3C_1\sin\theta + {}_3C_2\sin 2\theta + {}_3C_3\sin 3\theta\right)$

$$= 1 + {}_3C_1(\cos\theta + i\sin\theta) + {}_3C_2(\cos 2\theta + i\sin 2\theta) + {}_3C_3(\cos 3\theta + i\sin 3\theta)$$

$$= 1 + {}_3C_1 e^{i\theta} + {}_3C_2 e^{2i\theta} + {}_3C_3 e^{3i\theta}$$

$$= 1 + {}_3C_1 e^{i\theta} + {}_3C_2 (e^{i\theta})^2 + {}_3C_3 (e^{i\theta})^3$$

$$= \left(1 + e^{i\theta}\right)^3$$

10. $C - iS = 1 - \frac{\cos\theta}{2} + \frac{\cos 2\theta}{2^2} - \frac{\cos 3\theta}{2^3} - i\left(\frac{\sin\theta}{2} - \frac{\sin 2\theta}{2^2} + \frac{\sin 3\theta}{2^3}\right) + \ldots..$

$$= 1 - \frac{\cos\theta + i\sin\theta}{2} + \frac{\cos 2\theta + i\sin 2\theta}{2^2} - \frac{\cos 3\theta + i\sin 3\theta}{2^3} + \ldots..$$

$$= 1 - \frac{e^{i\theta}}{2} + \frac{e^{2i\theta}}{2^2} - \frac{e^{3i\theta}}{2^3} + \ldots..$$

$$= 1 - \frac{e^{i\theta}}{2} + \left(\frac{e^{i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta}}{2}\right)^3 + \ldots..$$