Section 2: Applications of de Moivre's theorem

Notes and Examples

These notes contain subsections on:

- Using de Moivre's theorem to find trigonometric identities
- The exponential form of a complex number
- Using complex numbers to sum real series

Using de Moivre's theorem to find trigonometric identities

de Moivre's theorem can be used to give a multiple angle formula in terms of powers and to express powers of sine and cosine in terms of multiple angles. This can be very useful in integration.

In the example below, the identities $\sin n\theta = \dfrac{z^n - z^{-n}}{2\mathrm{i}}$ and $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$ are used frequently. You should be happy with these identities and be able to derive them as follows:

$$z^n = \cos n\theta + \mathrm{i}\sin n\theta$$
$$z^{-n} = \cos n\theta - \mathrm{i}\sin n\theta$$

Adding: $z^n + z^{-n} = 2\cos n\theta \quad \Rightarrow \cos n\theta = \dfrac{z^n + z^{-n}}{2}$

Subtracting: $z^n - z^{-n} = -2\mathrm{i}\sin n\theta \quad \Rightarrow \sin n\theta = \dfrac{z^n - z^{-n}}{2\mathrm{i}}$

Example 1

(i) Express $\sin^6\theta$ in terms of multiple angles

(ii) Hence find $\int \sin^6\theta \, \mathrm{d}\theta$.

Solution

(putting $n = 1$)

(i) $z = \cos\theta + \mathrm{i}\sin\theta \Rightarrow \sin n\theta = \dfrac{z^n - z^{-n}}{2\mathrm{i}} \Rightarrow 2\mathrm{i}\sin\theta = z - z^{-1}$.

So $(2\mathrm{i})^6 \sin^6\theta = \left(z - z^{-1}\right)^6$

$$= z^6 - 6z^5z^{-1} + 15z^4z^{-2} - 20z^3z^{-3} + 15z^2z^{-4} - 6zz^{-5} + z^{-6}$$
$$= z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}$$
$$= z^6 + z^{-6} - 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) - 20$$
$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

(using $z^n + z^{-n} = 2\cos n\theta$ for $n = 6$, 4 and 2 respectively)

Therefore: $-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$.

$$\Rightarrow \qquad \sin^6\theta = \frac{20 - 2\cos 6\theta + 12\cos 4\theta - 30\cos 2\theta}{64}$$
$$= \frac{10 - \cos 6\theta + 6\cos 4\theta - 15\cos 2\theta}{32}$$

(ii) $$\int \sin^6\theta \, \mathrm{d}\theta = \int \left(\frac{10 - \cos 6\theta + 6\cos 4\theta - 15\cos 2\theta}{32} \right) \mathrm{d}\theta$$
$$= \frac{1}{32}\left(10\theta - \frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta - \frac{15}{2}\sin 2\theta \right) + c$$
$$= \frac{1}{192}\left(60\theta - \sin 6\theta + 9\sin 4\theta - 45\sin 2\theta \right) + c$$

The exponential form of a complex number

The idea of raising a number to a power *which is a natural number* is very familiar to you. It just means multiply that number by itself the specified number of times. For negative numbers and for rational numbers you have interpretations too (using reciprocals and roots). The idea of raising a number to the power of a complex number does not seem like a natural thing to do. Whichever way this is interpreted, it should have properties which are consistent with the ones that you are familiar with for rational numbers.

Using the Maclaurin expansion for e^x and putting $x = \mathrm{i}\theta$ leads to the result

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta .$$

This may seem a surprising result, but try not to let any concerns you have about the reason for making this definition get in the way of your ability to do questions. Once you have seen the power of this definition you will forget any worries you had about where it came from (this is also true of its acceptance historically in mathematics).

A particularly interesting result comes from putting $\theta = \pi$. This gives

$$\mathrm{e}^{\mathrm{i}\pi} = \cos\pi + \mathrm{i}\sin\pi$$
$$= -1$$

This famous equation is sometimes called Euler's equation. It links the irrational numbers π and e, along with the imaginary number i, giving the simple result of -1.

Using $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, you can express a complex number $z = r(\cos\theta + \mathrm{i}\sin\theta)$ as $z = \mathrm{e}^{\mathrm{i}\theta}$. This is called the exponential form of the complex number. Like the modulus-argument form, it expresses the complex number in terms of its modulus and argument, but the exponential form is more compact and can be easy to work with in some situations, since you can easily apply the laws of indices.

Sometimes you might be asked to prove trig identities which have been given in terms of the definition $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$. Here is an example of this.

Example 1

Prove that $1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta=-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\sin\theta$.

Solution

$$\begin{aligned}1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta &= 1-(\cos\theta+\mathrm{i}\sin\theta)\cos\theta\\ &= 1-\cos^2\theta-\mathrm{i}\sin\theta\cos\theta\\ &= \sin^2\theta-\mathrm{i}\sin\theta\cos\theta\\ &= \sin\theta(\sin\theta-\mathrm{i}\cos\theta)\\ &= \sin\theta\left[-\mathrm{i}(\cos\theta+\mathrm{i}\sin\theta)\right]\\ &= -\mathrm{i}\sin\theta\mathrm{e}^{\mathrm{i}\theta}\end{aligned}$$

as required.

Expand the left hand side using the definition of $\mathrm{e}^{\mathrm{i}\theta}$.

Since the right hand side has a factor of $\sin\theta$, it seems sensible to replace $1-\cos^2\theta$ with $\sin^2\theta$.

Other examples of identities like this one include

$$(1+\mathrm{e}^{\mathrm{i}\theta})(1+\mathrm{e}^{-\mathrm{i}\theta})=2+2\cos\theta$$
$$(2+\mathrm{e}^{\mathrm{i}\theta})(2+\mathrm{e}^{-\mathrm{i}\theta})=5+4\cos\theta$$
$$1+\mathrm{e}^{\mathrm{i}\theta}=2\mathrm{e}^{\mathrm{i}\theta/2}\cos\frac{\theta}{2}$$
$$1-\mathrm{e}^{\mathrm{i}\theta}=-2\mathrm{i}\mathrm{e}^{\mathrm{i}\theta/2}\sin\frac{\theta}{2}$$

Try proving some of these for yourself. To do this you will need to use identities such as $1+\cos2\alpha=2\cos^2\alpha$ and $\sin2\alpha=2\cos\alpha\sin\alpha$.

Using complex numbers to sum real series.

Consider the sum

$$1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos2\theta+\ldots+\binom{n}{n-1}\cos\left[(n-1)\theta\right]+\binom{n}{n}\cos n\theta.$$

This sum is not an arithmetic series, it is not a geometric series, nor is it a binomial series. However, in Example 2 below, complex numbers will be used to show that it is equal to

$$\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}.$$

Similarly, it is not immediately clear that the series

$$\frac{\sin\theta}{2}-\frac{\sin2\theta}{2^2}+\frac{\sin3\theta}{2^3}-\frac{\sin4\theta}{2^4}+\ldots$$

can be dealt with using the techniques that we have learnt so far. It is shown in Example 3, using complex numbers, that the sum of this series is

$$\frac{2\sin\theta}{5+4\cos\theta}$$

In both examples, the strategy is the same:

- introduce some complex terms so that the given summation becomes the real or imaginary part of an arithmetic, geometric or binomial series. Then, using the familiar formulae for these types of sums, find an expression for the (usually complex) value of this new summation. In both examples below, De Moivre's Theorem, that $(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta$, or $\left(\mathrm{e}^{\mathrm{i}\theta}\right)^n=\mathrm{e}^{\mathrm{i}n\theta}$, is used.

- manipulate this expression to express it as a single complex number. In both of the examples below, some of this algebraic manipulation is dealt with in the first part of the question.

- equate real or imaginary parts to give the value of the original summation.

Example 2

(i) Using the identities $1+\cos 2\alpha=2\cos^2\alpha$ and $\sin 2\alpha=2\cos\alpha\sin\alpha$, prove that

$$1+\mathrm{e}^{\mathrm{i}\theta}=2\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\cos\frac{\theta}{2}.$$

(ii) Let $C=1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\ldots+\binom{n}{n-1}\cos\left[(n-1)\theta\right]+\binom{n}{n}\cos n\theta$.

By considering $C+\mathrm{i}S$, where

$$S=\binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\ldots+\binom{n}{n-1}\sin\left[(n-1)\theta\right]+\binom{n}{n}\sin n\theta$$

show that $C=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$.

The identity $1+\cos 2\alpha=2\cos^2\alpha$ gives $1+\cos\theta=2\cos^2\frac{\theta}{2}$

Solution

(i) $1+\mathrm{e}^{\mathrm{i}\theta}=1+\cos\theta+\mathrm{i}\sin\theta$

$$=2\cos^2\frac{\theta}{2}+2\mathrm{i}\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$=2\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2}$$

$$=2\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\cos\frac{\theta}{2}$$

The identity $\sin 2\alpha=2\cos\alpha\sin\alpha$ gives $\sin\theta=2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$

(ii) $$C+\mathrm{i}S=\left[1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\ldots+\binom{n}{n-1}\cos[(n-1)\theta]+\binom{n}{n}\cos n\theta\right]$$
$$+\mathrm{i}\left[\binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\ldots+\binom{n}{n-1}\sin[(n-1)\theta]+\binom{n}{n}\sin n\theta\right]$$
$$=1+\binom{n}{1}(\cos\theta+\mathrm{i}\sin\theta)+\binom{n}{2}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\ldots$$
$$+\binom{n}{n-1}\left(\cos[(n-1)\theta]+\mathrm{i}\sin[(n-1)\theta]\right)+\binom{n}{n}(\cos n\theta+\mathrm{i}\sin n\theta)$$
$$=1+\binom{n}{1}(\cos\theta+\mathrm{i}\sin\theta)+\binom{n}{2}(\cos\theta+\mathrm{i}\sin\theta)^2+\ldots$$
$$+\binom{n}{n-1}(\cos\theta+\mathrm{i}\sin\theta)^{n-1}+\binom{n}{n}(\cos\theta+\mathrm{i}\sin\theta)^n$$
$$=1+\binom{n}{1}\mathrm{e}^{\mathrm{i}\theta}+\binom{n}{2}\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2+\ldots+\binom{n}{n-1}\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{n-1}+\binom{n}{n}\left(\mathrm{e}^{\mathrm{i}\theta}\right)^n$$

using de Moivre's theorem

$$=\left(1+\mathrm{e}^{\mathrm{i}\theta}\right)^n$$

This is just the standard result $(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n$, with $x=\mathrm{e}^{\mathrm{i}\theta}$

Using the answer to part (i) gives
$$C+\mathrm{i}S=\left(1+\mathrm{e}^{\mathrm{i}\theta}\right)^n$$
$$=\left(2\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\cos\frac{\theta}{2}\right)^n$$
$$=2^n\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}\right)^n$$
$$=2^n\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)\left(\cos\frac{\theta}{2}\right)^n.$$

Equating the real parts of the equation above gives $C=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$.

Example 3

(i) Show that $\left(2+\mathrm{e}^{\mathrm{i}\theta}\right)\left(2+\mathrm{e}^{-\mathrm{i}\theta}\right)=5+4\cos\theta$.

(ii) Let $S=\frac{\sin\theta}{2}-\frac{\sin 2\theta}{2^2}+\frac{\sin 3\theta}{2^3}-\frac{\sin 4\theta}{2^4}+\ldots\ldots$

By considering $C-\mathrm{i}S$ where $C=1-\frac{\cos\theta}{2}+\frac{\cos 2\theta}{2^2}-\frac{\cos 3\theta}{2^3}+\frac{\cos 4\theta}{2^4}-\ldots$, show that $S = \dfrac{2\sin\theta}{5 + 4\cos\theta}$.

Solution

(i) $$\begin{aligned}\left(2+e^{i\theta}\right)\left(2+e^{-i\theta}\right) &= 4 + 2e^{i\theta} + 2e^{-i\theta} + e^{i\theta}e^{-i\theta} \\ &= 5 + 2\left(e^{i\theta} + e^{-i\theta}\right) \\ &= 5 + 2\left[\left(\cos\theta + i\sin\theta\right) + \left(\cos(-\theta) + i\sin(-\theta)\right)\right] \\ &= 5 + 2\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right) \\ &= 5 + 2\times(2\cos\theta) \\ &= 5 + 4\cos\theta.\end{aligned}$$

(ii) $$\begin{aligned}C - iS &= 1 - \frac{\cos\theta}{2} - i\frac{\sin\theta}{2} + \frac{\cos 2\theta}{2^2} + i\frac{\sin 2\theta}{2^2} - \frac{\cos 3\theta}{2^3} - i\frac{\sin 3\theta}{2^3} + \ldots \\ &= 1 - \left(\frac{\cos\theta + i\sin\theta}{2}\right) + \left(\frac{\cos 2\theta + i\sin 2\theta}{2^2}\right) - \left(\frac{\cos 3\theta + i\sin 3\theta}{2^3}\right) + \ldots \\ &= 1 - \left(\frac{\cos\theta + i\sin\theta}{2}\right) + \frac{\left(\cos\theta + i\sin\theta\right)^2}{2^2} - \frac{\left(\cos\theta + i\sin\theta\right)^3}{2^3} + \ldots \\ &= 1 - \left(\frac{\cos\theta + i\sin\theta}{2}\right) + \left(\frac{\cos\theta + i\sin\theta}{2}\right)^2 - \left(\frac{\cos\theta + i\sin\theta}{2}\right)^3 + \ldots \\ &= 1 - \frac{e^{i\theta}}{2} + \left(\frac{e^{i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta}}{2}\right)^3 + \ldots \\ &= \frac{1}{1 - \left(-\frac{e^{i\theta}}{2}\right)} \\ &= \frac{2}{2 + e^{i\theta}}\end{aligned}$$

This is just a geometric series with first term 1 and common ratio $-\dfrac{e^{i\theta}}{2}$

using the standard result that the sum of an infinite geometric progression with first term a and common ration r is $\dfrac{a}{1-r}$, with a = 1 and $r = -\dfrac{e^{i\theta}}{2}$. Note $|r| < 1$ is required for the series to converge, check you are happy that this is the case for $r = -\dfrac{e^{i\theta}}{2}$.)

Using part (i):

$$C - \mathrm{i}S = \frac{2}{2+\mathrm{e}^{\mathrm{i}\theta}}$$
$$= \left(\frac{2}{2+\mathrm{e}^{\mathrm{i}\theta}}\right)\left(\frac{2+\mathrm{e}^{-\mathrm{i}\theta}}{2+\mathrm{e}^{-\mathrm{i}\theta}}\right).$$
$$= \frac{4+2\mathrm{e}^{-\mathrm{i}\theta}}{5+4\cos\theta}$$
$$= \frac{4+2\cos\theta - 2\mathrm{i}\sin\theta}{5+4\cos\theta}$$

Equating imaginary parts gives $S = \dfrac{2\sin\theta}{5+4\cos\theta}$.